Complex Geometry

Automorphisms and deformations

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November 19, 2026

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Automorphisms of bundles

Gauge transformations

For a smooth complex bundle E, its group of symmetries (gauge group) is

$$\mathcal{G} = \mathcal{A}^0(X, \mathrm{GL}(E)),$$

where $\mathrm{GL}(E) \subset \mathrm{End}(E)$ consists of invertible fibre endomorphisms.

It has the structure of an *infinite dimensional* (Fréchet) Lie group.

Its associated Lie algebra is $\mathfrak{G} = \mathcal{A}^0(X, \mathrm{End}(E))$. It acts on Dolbeault operators by conjugation

$$\bar{\partial} \longmapsto g^{-1}\bar{\partial}(g\,\cdot).$$

The **holomorphic** automorphisms of $(E, \bar{\partial}_E)$ are precisely the stabilizer of $\bar{\partial}_E$:

$$\mathrm{Aut}(E) \coloneqq \{g \in \mathcal{G} \mid g^{-1}\bar{\partial}_E g = \bar{\partial}_E\}$$

Infinitesimal bundle automorphisms

Write $g_t = \mathrm{Id} + ta + O(t^2)$ and take $v \in E$. Then

$$g_t^{-1}\bar{\partial}_E(g_t v) = \bar{\partial}_E v + t(\bar{\partial}_{\mathrm{End}\, E} a)v + O(t^2).$$

Preserving the holomorphic structure to first order means

$$\bar{\partial}_{\mathrm{End}\, E} a = 0.$$

The Lie algebra of $\mathrm{Aut}(E)$ is the finite dimensional Lie algebra $H^0(X, \mathrm{End}\, E)$.

The induced complex structure makes it a complex Lie algebra:

$$J[a, b] = [Ja, b] = [a, Jb].$$

$\mathrm{Aut}(E)$ **is a complex Lie group**

Direct sums and simple bundles

$\mathbb{C}^* \subseteq \mathrm{Aut(E)}$ by scalar multiplication.

If $E = E_1 \oplus E_2$, independent scalar multiplication on each summand gives $(\mathbb{C}^\times)^2 \subset \mathrm{Aut}(E)$.

Simplicity

E is simple if $H^0(X, \operatorname{End} E) = \mathbb{C}\mathrm{Id}$, equivalently $H^0(X, \operatorname{End}_0 E) = 0$.

A simple bundle is indecomposable. The converse is not necessarily true: non-scalar endomorphisms can be nilpotent without producing a splitting. (think non-diagonal spectral decompositions)

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Automorphisms of manifolds

Holomorphic vector fields

For the remainder, (X, J) is The gauge group is $\mathcal{G} = \mathrm{Diff}(X)$. Its Lie algebra is

$$\mathfrak{G} = \mathcal{A}^0(X, TX) = \mathfrak{X}(X) .$$

The automorphism group is

$$\mathrm{Aut}(X) = \{f : X \to X : f \text{ is biholomorphic}\}.$$

Automorphism-group theorem

For compact complex X, this is a finite-dimensional complex Lie group with

$$\mathrm{Lie}\,\mathrm{Aut}(X) = H^0(X, T^{1,0}X).$$

The real part of a holomorphic vector field generates a flow preserving J. Compactness guarantees the real flow is complete.

Linearizing preservation of the complex structure

For a flow $\Phi_t = \mathrm{Id} + tv$ generated by a real vector field V,

$$\Phi_t^* J = J \quad \Longrightarrow \quad \mathcal{L}_V J = 0.$$

Writing $V = v + \bar{v}$ with $v \in \mathcal{A}^0(T^{1,0}X)$, this is equivalent to

$$\bar{\partial} v = 0.$$

Brackets of holomorphic vector fields remain holomorphic and are complex bilinear, so $H^0(X, T^{1,0}X)$ is again a complex Lie algebra.

Automorphisms of projective space

Linear transformations descend to projective automorphisms with scalar kernel:

$$1 \to \mathbb{C}^\times \to GL(n+1,\mathbb{C}) \to \mathrm{PGL}(n,\mathbb{C}) \to 1.$$

with $\mathrm{PGL}(n,\mathbb{C}) \subseteq \mathrm{Aut}(\mathbb{CP}^n)$. Every automorphism preserves the positive generator $\mathcal{O}(1)$ of $\mathrm{Pic}(\mathbb{CP}^n)$ and therefore acts projectively on its $(n+1)$-dimensional section space. Thus

$$\mathrm{Aut}(\mathbb{CP}^n) \cong PGL(n+1,\mathbb{C}), \qquad \dim = (n+1)^2 - 1.$$

The projective line

$$\mathrm{Aut}(\mathbb{CP}^1) = PGL(2,\mathbb{C}), \qquad z \longmapsto \frac{az+b}{cz+d}.$$

The tangent bundle is $\mathcal{O}(2)$, whose sections on the affine chart are

$$\partial_z, \qquad z\partial_z, \qquad z^2\partial_z.$$

A Möbius transformation is determined by its action on three distinct points. This matches the three-dimensional Lie algebra.

Automorphisms of complex tori

For $T = \mathbb{C}^n/\Lambda$, the identity component has dimension n.It consists of translation.

Compose an automorphism with a translation so that it fixes the origin. Its lift $F : \mathbb{C}^n \to \mathbb{C}^n$ has lattice-periodic derivative.

That derivative descends to a holomorphic matrix-valued function on compact T, so it is constant. Hence $F(z) = g\,z$ with $g \in \mathrm{GL(n, \mathbb{C})}$

$$\mathrm{Aut}(T) = T \rtimes \{g \in GL(n, \mathbb{C}) \mid g\Lambda = \Lambda\}.$$

Riemann surfaces

For a compact genus-g curve,

$$\deg T_C = 2 - 2g.$$

If $g \geq 2$, the tangent bundle has negative degree, so it has no non-zero holomorphic sections. Its automorphism group is finite.

$$\left|\operatorname{Aut}(C)\right| \leq 84(g-1)$$

This follows from what we have seen and Poincaré-Lefschetz theory.

Positive canonical bundle

If K_X is positive, Serre duality gives

$$H^0(X, T^{1,0}X)^* \cong H^n(X, \Omega^1_X \otimes K_X).$$

Kodaira–Akizuki–Nakano vanishing applies since $1 + n > n$. Thus $H^0(X, T^{1,0}X) = 0$.

The automorphism group is discrete; in fact it is **finite**.

More generally, smooth projective manifolds of general type ($\kappa = \dim X$) have finite automorphism group. Positivity of K_X is a stronger condition.

Trivial canonical bundle

If $K_X \cong \mathcal{O}_X$, choose a nowhere-zero holomorphic volume form Ω. Contraction gives

$$T^{1,0}X \xrightarrow{\lrcorner\Omega} \Omega_X^{n-1}.$$

Therefore

$$h^0(X, T^{1,0}X) = h^{n-1,0}(X).$$

- For a complex torus $h^{n-1,0} = n$, corresponding to translations.
- For an **irreducible** Calabi–Yau manifold, $h^{n-1,0} = 0$

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Deformation theory of holomorphic bundles

The affine space of Dolbeault operators

Fix a smooth bundle E and a holomorphic structure $\bar{\partial}_E$.

Any other $(0,1)$-operator with the Leibniz rule is

$$D'' = \bar{\partial}_E + \alpha, \qquad \alpha \in \mathcal{A}^{0,1}(\mathrm{End}\, E).$$

The subset defining holomorphic structures is cut out by the zero locus map:

$$\begin{aligned} F : \mathcal{A}^{0,1}((\mathrm{End}\, E) &\to \mathcal{A}^{0,2}((\mathrm{End}\, E) \\ \alpha &\mapsto (D'')^2 \end{aligned}$$

Idea

Use some form of Implicit Function Theorem to characterise the vanishing locus of F.

The Maurer–Cartan equation and first-order deformations modulo gauge

The integrability condition is

$$(\bar\partial_E + \alpha)^2 = \bar\partial_{\mathrm{End}\,E}\alpha + \alpha \wedge \alpha.$$

For $\alpha(t) = t\alpha_1 + t\alpha_2 + O(t^3)$, the linear equation is

$$\bar\partial \alpha_1 = 0.$$

The infinitesimal gauge change $g_t = \mathrm{Id} + tu + O(t^2)$ replaces α_1 by $\alpha_1 + \bar\partial u$. Therefore first-order deformations modulo gauge are

$$\frac{\ker \bar\partial : \mathcal{A}^{0,1}(\mathrm{End(E)}) \to \mathcal{A}^{0,2}(\mathrm{End(E)})}{\mathrm{im}\,\bar\partial : \mathcal{A}^{0}(\mathrm{End(E)}) \to \mathcal{A}^{0,1}(\mathrm{End(E)})} \cong H^1(X, \mathrm{End}\,E).$$

Second-order obstructions

At order t^2, we need to solve

$$\bar{\partial}\alpha_2 = -\tfrac{1}{2}[\alpha_1, \alpha_1].$$

The right-hand side is $\bar{\partial}$-closed. Its class in $H^2(X, \operatorname{End} E)$ must vanish for the first-order deformation to extend.

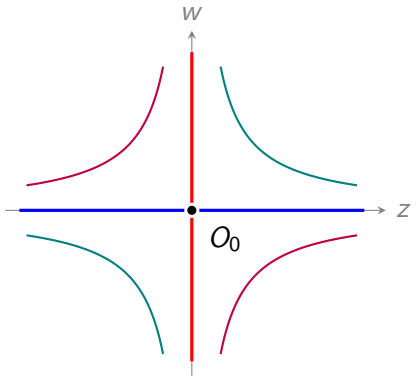
The map F is **Fredholm**, so we have a well-defined **obstruction map**:

$$\Theta : H^1(X, \operatorname{End}(E)) \to H^2(X, \operatorname{End}(E)) \,.$$

By the **implicit function theorem**, the moduli space is modelled on $\Theta^{-1}(0)/\operatorname{Aut}(E)$.

The IFT gives *a* deformation family, but not a **smooth moduli space**.

Non-Hausdorff phenomena



Real slice of $\mathbb{C}^2$

Consider the action of $\mathbb{C}^*$ on $\mathbb{C}^2$ given by

$$\lambda \cdot (z, w) = (\lambda z, \lambda^{-1} w).$$

Non-zero axis orbits have the origin and axes in their closures.

Their images cannot be separated in the naive orbit space if $wz = 0$.

$\implies$ Quotient is non-Hausdorff

Slope stability

On compact Kähler (X, ω), define the notion of **slope**:

$$\mu_\omega(E) = \frac{1}{\operatorname{rk} E} \int_X c_1(E) \wedge \frac{\omega^{n-1}}{(n-1)!}.$$

E is slope stable if every **coherent** subsheaf $0 \neq F \subset E$ with $0 < \operatorname{rk} F < \operatorname{rk} E$ satisfies

$$\mu(F) < \mu(E)$$

Sheaves vs bundles

The use of coherent subsheaves is essential!

$$\text{Slope stability } + H^2(X, \operatorname{End}(E)) = 0 \implies \text{Local smooth moduli .}$$

Line bundles as a deformation example

For a line bundle, $\operatorname{End} L = \mathcal{O}_X$ and the bracket vanishes.

The local deformation space is smooth with tangent $H^1(X, \mathcal{O}_X)$... which we already knew from the exponential sequence!

Globally, holomorphic structures with fixed topology form a torsor under

$$\operatorname{Pic}^0(X) = H^1(X, \mathcal{O}_X) / \operatorname{im} H^1(X, \mathbb{Z}).$$

For Kähler X, it is a torus complex of dimension $b_1/2$.

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Deformations of complex manifolds

Variations of an almost complex structure

Differentiating $J_t^2 = -\mathrm{Id}$ at $t = 0$ gives

$$\dot{J}J + J\dot{J} = 0.$$

Thus $\dot{J}$ exchanges the $(1,0)$ and $(0,1)$ spaces. A nearby structure can be described by a small Beltrami differential

$$\varphi \in \mathcal{A}^{0,1}(T^{1,0}X), \qquad T^{0,1}_{\varphi} = \{v + \varphi(v) : v \in T^{0,1}X\}.$$

Smallness ensures the graph is transverse to its complex conjugate, so it defines an almost complex structure.

Integrability and the vector-field bracket

In local coordinates, the new $(0,1)$ fields are

$$\frac{\partial}{\partial \bar{z}_j} + \sum_k \varphi_{\bar{j}}^{\ k} \frac{\partial}{\partial z_k}.$$

Their brackets close in the new distribution precisely when

$$\boxed{\bar{\partial}\varphi + \tfrac{1}{2}[\varphi, \varphi] = 0.}$$

Here the bracket comes from the Lie bracket of vector fields, rather than pointwise endomorphism composition.

The Kodaira–Spencer tangent space

Linearizing integrability gives $\bar{\partial}\varphi_1 = 0$.

Infinitesimal changes induced by diffeomorphisms are $\bar{\partial}v$, with $v \in \mathcal{A}^{0,0}(T^{1,0}X)$.

Therefore, we get

Infinitesimal automorphisms	$H^0(X, T^{1,0}X)$
First-order deformations	$H^1(X, T^{1,0}X)$.
Obstruction space	$H^2(X, T^{1,0}X)$

As before, Fredholm theory allows us to construct a semi-universal versal family

Curves: deformation dimensions

For a compact genus-g curve C, $H^2(T^{1,0}C) = 0$

Genus	$h^0(T_C)$	$h^1(T_C)$	Local deformation dimension
0	3	0	0
1	1	1	1
$g \geq 2$	0	$3g-3$	$3g-3$

For $g \geq 2$, marked complex structures form Teichmüller space $\mathcal{T}(C)$, a complex manifold of dimension $3g-3$.

The mapping class group acts, and the unmarked moduli space is its quotient. Finite automorphism groups give finite local stabilizers.

Fano manifolds are unobstructed

If K_X^{-1} is positive, then

$$T^{1,0}X \cong \Omega_X^{n-1} \otimes K_X^{-1}.$$

Kodaira–Akizuki–Nakano vanishing gives

$$H^2(X, T^{1,0}X) = H^2(X, \Omega_X^{n-1} \otimes K_X^{-1}) = 0,$$

since $(n-1)+2 > n$. Thus Fano manifolds have smooth Kuranishi germs, although automorphisms may still affect their coarse moduli spaces. The same vanishing theorem gives no automatic conclusion for $H^1(TX)$.

Complex tori

For a complex torus T^n, the tangent bundle is trivial.

Harmonic forms have constant coefficients for a flat metric:

$$h^q(T^n, T^{1,0}T^n) = n\binom{n}{q}.$$

The tangent space has dimension n^2, and the obstruction group is non-zero for $n \geq 2$.

Constant Beltrami differentials have zero bracket, so they integrate directly.

$\implies$ Universal family has maximal dimension n^2 despite the non-zero obstruction space.

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Calabi–Yau deformations and periods

Contraction with a holomorphic volume form

Let X be compact Kähler with $K_X \cong \mathcal{O}_X$.

Choose a nowhere-zero holomorphic n-form Ω.

$$\mathcal{A}^{0,q}(T^{1,0}X) \xrightarrow{\lrcorner\Omega} \mathcal{A}^{n-1,q}(X).$$

This intertwines Dolbeault operators, giving

$$H^q(X, T^{1,0}X) \cong H^{n-1,q}(X).$$

Thus deformations have dimension $h^{n-1,1}$ infinitesimally.

The potential obstruction space is $H^{n-1,2}$, which need not vanish.

The Tian–Todorov lemma and first obstructions

Assume (X, J, ω) is Kähler. We have the key identity

$$[\alpha, \beta] \lrcorner \Omega = \partial\big(\alpha \lrcorner (\beta \lrcorner \Omega)\big) - \big\langle \partial(\alpha \lrcorner \Omega), \Omega \big\rangle\, \beta \lrcorner \Omega + \big\langle \partial(\beta \lrcorner \Omega), \Omega \big\rangle\, \alpha \lrcorner \Omega.$$

Each class in $H^1(TX)$ is represented by a $(n-1,1)$ harmonic form

$$\partial(\alpha \lrcorner \Omega) = 0 = \bar\partial(\alpha \lrcorner \Omega)$$

By the key identity, the commutator $[\alpha, \beta] \lrcorner \Omega$ is ∂-exact.

By the $\partial\bar\partial$-lemma, $[\alpha, \beta] \lrcorner \Omega$ is $\bar\partial$-exact, so

$$[\cdot, \cdot] : H^1(X, TX) \times H^1(X, TX) \to H^2(X, TX) \quad \text{is zero.}$$

This proves vanishing of the quadratic obstruction.

Higher orders and Bogomolov–Tian–Todorov

Seek a solution $\varphi(t) = \sum_{k \geq 1} t^k \varphi_k$ of the Maurer–Cartan Equation. Recursively,

$$\bar{\partial}\varphi_k = -\frac{1}{2} \sum_{i+j=k} [\varphi_i, \varphi_j].$$

The key identity makes the contracted right-hand side ∂-exact.

The inductive integrability identity makes it $\bar{\partial}$-closed.

The $\partial\bar{\partial}$-lemma grantees it is $\bar{\partial}$-exact.

Theorem (Bogomolov–Tian–Todorov)

The Kuranishi germ is smooth of dimension $h^{n-1,1}$.

Thank you for attending the course!

Questions and feedback are exptremely welcome.

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