Complex Geometry

Kodira embedding and blow-ups

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Hard Lefschetz and Lefschetz $(1,1)$-theorem

The Kähler and Akizuki-Nakano identities

On a Kähler manifold (X, g, J, ω) we have the following relations between Dolbeault and the Lefschetz operators

$$\boxed{[L, \bar{\partial}^*] = -i\partial, \qquad [L, \partial^*] = i\bar{\partial}, \qquad [\Lambda, \bar{\partial}] = -i\partial^*, \qquad [\Lambda, \partial] = i\bar{\partial}^*.}$$

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Lemma (Akizuki–Nakano)

On a hermitian bundle (E, h) with Chern connection D_E we have

$$[\Lambda, \bar{\partial}_E] = -i(D_E^{1,0})^*$$

where D_E^* is the Chern connection on the dual bundle (E^*, h).

The $\partial\bar{\partial}$-lemma

Let α be a d-closed (p, q)-form on a compact Kähler manifold. The following are equivalent:

1. α is d-exact.
2. α is ∂-exact.
3. α is $\bar{\partial}$-exact.
4. α is $\partial\bar{\partial}$-exact.
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Proof 5 $\implies$ 4 (Idea).

Let G be the Green operator for $\Delta_{\bar{\partial}}$. Since α is d-closed,

$$\alpha = \Delta_{\bar{\partial}} G\alpha = \bar{\partial}\bar{\partial}^* G\alpha = i\bar{\partial}\partial\Lambda G\alpha = -i\partial\bar{\partial}\Lambda G\alpha.$$

Thus α is $\partial\bar{\partial}$-exact. □

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Theorem

There is a cohomological primitive decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{j \geq 0} L^j H^{k-2j}_{\text{prim}}(X, \mathbb{C}).$$

Lefschetz $(1,1)$-theorem

On a compact Kähler manifold, the image of $c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ contains the lattice

$$H^{1,1}(X, \mathbb{Z}) := \mathrm{im}\left(H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C})\right) \cap H^{1,1}(X, \mathbb{C}).$$

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Proof (Idea).

Combine the cohomology decomposition $H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X,\mathbb{C})$ with the exponential sequence, $\mathrm{Pic}(X) \to H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X)$. □

2

Positivity and vanishing

Positive line bundles

A real $(1,1)$-form α is positive whenever, for $0 \neq v$

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Example

The bundle $\mathcal{O}(1)$ is positive, with $\omega_{\mathcal{O}(1)} = \omega_{FS}$.

Bochner technique

From the Akizuki-Nakano-Bochner formula, we get

$$\Delta_{\bar{\partial}_L} = \Delta_{\partial_L} + [iF_L \wedge (\cdot), \Lambda].$$

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Let L be positive and choose $\omega = \frac{i}{2\pi} F_L$. On (p, q)-forms,

$$[iF_L, \Lambda] = 2\pi [L, \Lambda] = 2\pi (p + q - n) \mathrm{Id}.$$

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Example

$$H^q(\mathbb{CP}^n, \mathcal{O}(m)) \cong \begin{cases} \mathbb{C}[z_0, \ldots, z_n]_m, & q = 0, \ m \geq 0, \\ \mathbb{C}[z_0, \ldots, z_n]^*_{-m-n-1}, & q = n, \ m \leq -n-1, \\ 0, & \text{otherwise.} \end{cases}$$

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Let A be positive line bundle and L any holomorphic line bundle. Then, for $k >> 0$,

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Proof (Idea).

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$$\mathcal{O}(D) = \mathcal{O}\big((s)\big) \otimes \mathcal{O}\big((t)\big)^k = L \otimes \mathcal{O}(k) \otimes \mathcal{O}(-k) \cong L \, .$$

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Replacing rational coefficients by integral coefficients gives a false statement.

3
Kodaira embedding

The embedding theorem

Theorem (Kodaira)

If X is compact complex and L is positive, then for all sufficiently large m the pluricanonical map of L^m gives a holomorphic embedding

$$\varphi_{L^m} : X \hookrightarrow \mathbb{CP}^N.$$

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Corollay

A complex manifold is projective **iff** it admits a **positive** line bundle.

Three geometric requirements

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Compactness turns an injective immersion into an embedding.

Evaluation maps and sufficient vanishing

For a pair of distinct points x, y, surjectivity of

$$H^0(X, L) \to L_x, \qquad H^0(X, L^m) \to L_x \oplus L_y$$

gives base-point freedom and point separation.

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$$H^1(X, \mathcal{I}_x \otimes L) = 0, \qquad H^1(X, \mathcal{I}_{\{x,y\}} \otimes L) = 0.$$

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is sufficient to prescribe the first derivatives of sections vanishing at x.

Together with a non-vanishing section, this makes $d\varphi_{L^m}$ injective at x.

The curve case

For a compact curve C of genus g, we need the vanishing of the line bundle cohomologies

$$H^1(C, L(-x)) = 0\,, \qquad H^1(C, L(-x-y)) = 0 \qquad \text{and} \qquad H^1(C, L(-2x)) = 0$$

Serre duality gives

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Every compact Riemann surface is projective.

4

From points to divisors: Blow-ups

The blow-up of a point

$$\mathrm{Bl}_0\,\mathbb{C}^n = \{(z,[\ell]) \in \mathbb{C}^n \times \mathbb{CP}^{n-1} : z \in \ell\} \longrightarrow \mathbb{C}^n$$

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$$E = \sigma^{-1}(0) \cong \mathbb{CP}^{n-1}.$$

The total space is the tautological bundle $\mathcal{O}_{\mathbb{CP}^{n-1}}(-1)$.

In the chart $\ell_1 \neq 0$, write

$$z_1 = t, \qquad z_j = tu_j \quad (j = 2, \ldots, n).$$

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The blow-down map is

$$\sigma(t, u_2, \ldots, u_n) = (t, tu_2, \ldots, tu_n), \qquad E = \{t = 0\}.$$

General blow-up and universal property

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Hence, the exceptional divisor of a blow-up has self-intersection -1.

Blow-ups convert points into divisors

For $n \geq 2$, blow up $x \in X$ and $x, y \in X$:

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The short exact sequences of **sheaves** on X

$$0 \to L \otimes \mathcal{I}_x \to L \to L_x \to 0$$
$$0 \to L \otimes \mathcal{I}_{x,y} \to L \to L_x \oplus L_y \to 0$$
$$0 \to L \otimes \mathcal{I}_x^2 \to L \otimes \mathcal{I}_x \to L_x \otimes (T^*_{\mathbb{C}}X)^{1,0} \to 0$$

correspond to the short exact sequences

$$0 \to \sigma^*L \otimes \mathcal{O}(-E) \to \sigma^*L \to \sigma^*L|_E \to 0$$
$$0 \to \sigma^*L \otimes \mathcal{O}(-E_1 - E_2) \to \sigma^*L \to \sigma^*L|_{E_1 \cup E_2} \to 0$$
$$0 \to \sigma^*L \otimes \mathcal{O}(-2E) \to \sigma^*L \otimes \mathcal{O}(-E) \to \big(\sigma^*L \otimes \mathcal{O}(-E)\big)\big|_E \to 0$$

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5

Kodaira embedding v2

Projectivity and the Kähler cone

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Rational classes are dense in this real vector space. Since the Kähler cone is open, it contains a rational class. □

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Repeat until X embeds in $\mathbb{CP}^{2n+1}$.