Complex Geometry

Hermitian and Kähler geometry, and positivity

Enric Solé-Farré

November 12, 2026

1
Hermitian manifolds

Hermitian metrics and fundamental forms

Consider a complex bundle $E \to M$.

A Hermitian metric is a section $h \in \Gamma(\overline{E} \otimes E)$ that satisfies

$$h(\overline{v}, w) = \overline{h(\overline{w}, v)} \qquad \text{and} \qquad h(\overline{v}, v) \geq 0.$$

Its real part $g = \frac{1}{2}\left(h + \overline{h}\right)$ is J-invariant and defines a pointwise inner product.

On $T_{\mathbb{C}}M$, the imaginary part $\omega = \frac{i}{2}\left(h - \overline{h}\right)$ defines a positive real $(1,1)$-form:

$$\omega(v, w) = g(Jv, w), \qquad \omega(v, Jv) = g(v, v) > 0.$$

Conversely, the $(1,1)$-form determines g and h, with $\mathrm{vol}_g = \frac{\omega^n}{n!}$.

A Hermitian manifold (M, h) is a complex manifold equipped with a hermitian metric

The Hodge star and Lefschetz operators

On an oriented real $2n$-dimensional inner-product space (V, g), we have

$$\alpha \wedge *\beta = g(\alpha, \beta)\,\mathrm{vol}_g, \qquad *^2 = (-1)^k \text{ on } \Lambda^k.$$

If we extend $*$ complex linearly, we get

$$* : \Lambda^{p,q} \longrightarrow \Lambda^{n-q,n-p} \qquad \overline{\alpha} \wedge *\beta = h(\alpha, \beta)\,\mathrm{vol}\,.$$

On a hermitian vector space (V, h, ω), we have the triple of operators

$$L\alpha := \omega \wedge \alpha, \qquad \Lambda := L^* = (-1)^k * L * \qquad \text{and} \qquad H|_{\Lambda^k} = (k-n)\mathrm{Id}.$$

They obey

$$[H, L] = 2L, \qquad [H, \Lambda] = -2\Lambda, \qquad [L, \Lambda] = H.$$

The exterior algebra $\Lambda^\bullet V^*$ carries an induced $\mathfrak{sl}_2$-representation.

Primitive forms and Lefschetz decomposition

We say a form α is primitive when $\Lambda\alpha = 0$, and write $P^{p,q} = \ker \Lambda \cap \Lambda^{p,q}$.

Every form decomposes uniquely and orthogonally as

$$\Lambda^{p,q} = \bigoplus_{j\geq 0} L^j P^{p-2j,q-2j}.$$

T

he vector spaces $P^{p,q}(\mathbb{C}^n) \subseteq \Lambda^{p,q}(\mathbb{C}^n)$ are irreducible $\mathrm{U}(n)$-representations

Weil's formula

Let α be primitive of type (p, q), with $k = p + q$. Then

$$*L^j\alpha = (-1)^{k(k+1)/2} i^{p-q} \frac{j!}{(n-k-j)!} L^{n-k-j}\alpha, \quad 0 \leq j \leq n-k.$$

2

Hodge theory

Harmonic representatives

Assume (X, h) is closed hermitian manifold.

Let $\bar\partial^*$ be the formal L^2-adjoint to $\bar\partial$. By Stokes', $\bar\partial^* = - * \partial *$. We define the Dolbeault Laplacian and $\bar\partial$-harmonic (p, q)-forms as

$$\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial \qquad\qquad \mathcal{H}^{p,q} := \ker \Delta_{\bar\partial} \cap \mathcal{A}^{p,q}$$

Since

$$\langle \Delta_{\bar\partial}\alpha, \alpha\rangle = \|\bar\partial\alpha\|^2 + \|\bar\partial^*\alpha\|^2,$$

harmonic forms satisfy $\bar\partial\alpha = \bar\partial^*\alpha = 0$, and minimize the norm in their $\bar\partial$-homology class,

if they exist.

Hodge decomposition

Elliptic theory gives the orthogonal decomposition

$$\mathcal{A}^{p,q}(X) = \operatorname{im} \bar{\partial} \oplus \mathcal{H}^{p,q}_{\bar{\partial}}(X) \oplus \operatorname{im} \bar{\partial}^*.$$

The space of $\bar{\partial}$-harmonic forms is finite dimensional and

Theorem

Every Dolbeault class has a unique harmonic representative:

$$\mathcal{H}^{p,q}_{\bar{\partial}}(X) \cong H^q(X, \Omega^p_X).$$

The same argument extends to holomorphic Hermitian bundles (E, h):

$$\mathcal{H}^{p,q}_{\bar{\partial}_E}(X, E) \cong H^q(X, \Omega^p_X \otimes \mathcal{E}).$$

But $\bar{\partial}^*_E$ is a priori only formal, since ∂_E makes no sense... or does it?

Serre duality

For a compact complex n-manifold and holomorphic bundle E, integration gives a perfect pairing

$$H^{p,q}_{\bar\partial_E}(X,E) \times H^{n-p,n-q}_{\bar\partial_E}(X,E^*) \to \mathbb{C}, \qquad ([\alpha],[\beta]) \mapsto \int_X \alpha \wedge \beta.$$

Equivalently,

$$H^q(X,\mathcal{E})^* \cong H^{n-q}(X,\mathcal{E}^* \otimes K_X).$$

The metric and Hodge star identify a non-zero harmonic representative with a dual representative having non-zero pairing.

3
Connections and curvature

Connections and local matrices

A connection is a $\mathbb{C}$-linear operator $\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ satisfying

$$\nabla(fs) = df \otimes s + f\nabla s.$$

In a local frame with sections written as,

$$\nabla = d + A, \qquad A \in \mathcal{A}^1(\operatorname{End} E).$$

Connections form an affine space modelled on $\mathcal{A}^1(\operatorname{End} E)$. On a complex manifold,

$$\nabla = \nabla^{1,0} + \nabla^{0,1}.$$

- A connection is compatible with the holomorphic structure if $\nabla^{0,1} = \bar{\partial}_E$.
- A connection is compatible with the metric if

$$dh(s,t) = h(\nabla s, t) + h(s, \nabla t)\ .$$

The curvature of a connection

A connection D extends to bundle-valued forms D by the graded Leibniz rule.

$$D(\alpha \wedge \beta) \coloneqq d\alpha \wedge \beta + (-1)^l \alpha \wedge D\beta \, .$$

with $\alpha \in \mathcal{A}^k(\mathbb{R})$ and $\beta \in \mathcal{A}^l(E)$.

We define the curvature as $F_\nabla = \nabla \circ \nabla : \mathcal{A}^k \to \mathcal{A}^{k+2}$.

$$F_D = D^2 \in \mathcal{A}^2(\operatorname{End} E) \qquad \text{a.k.a} \qquad F_D \text{ is tensorial.}$$

Locally,

$$F_D = dA + A \wedge A = dA + \tfrac{1}{2}[A, A] \qquad \text{(Maurer–Cartan equation).}$$

Bianchi identity

$$D(F_D) = D^3 = 0 \, .$$

The Chern connection

Existence and uniqueness

A holomorphic Hermitian bundle $(E, \bar\partial_E, h)$ has a unique holomorphic metric connection

$$\nabla^{0,1} = \bar\partial_E, \qquad \nabla h = 0 \quad (\iff d h(X,Y) - h(\nabla X, Y) + h(X, \nabla Y).$$

For a holomorphic frame and metric matrix h, in the column convention,

$$A^{1,0} = h^{-1}\partial h.$$

For a line bundle this is $A = \partial \log h$.

Its curvature is

$$F = \bar\partial(h^{-1}\partial h) \in \mathcal{A}^{1,1}(\operatorname{End} E).$$

The first Chern class from curvature

The Bianchi identity implies one may **extract** de Rham cohomology classes from the curvature F_∇.

For a line bundle, if $\nabla' = \nabla + a$, then $F_{\nabla'} = F_\nabla + da$, so the class is independent of the connection. In holomorphic frames,

$$F = \bar{\partial}\partial \log h = -\partial\bar{\partial} \log h.$$

Extend this to arbitrary vector bundles by $\mathrm{tr}(F_\nabla^k)$ – **Chern-Weil theory**

Theorem

$$c_1(L)_{\mathbb{R}} = \left[\frac{i}{2\pi} F_\nabla\right].$$

Divisors and the first Chern class

For a divisor D on a compact complex manifold,

$$c_1(\mathcal{O}(D))_{\mathbb{R}} = \mathrm{PD}[D].$$

For a smooth hypersurface Y and a closed $(2n-2)$-form γ,

$$\int_X \frac{iF}{2\pi} \wedge \gamma = \int_Y \gamma.$$

Proof (Idea).

A local equation for Y and Stokes' theorem reduce this to the residue of dz/z around a small normal circle. The claim follows from the Cauchy Integral formula. □

In particular, $c_1(\mathcal{O}(1))$ is the hyperplane class, as previously claimed.

4

Kähler metrics

The Kähler conditions

Let (X, g, J, ω) be almost Hermitian, with Levi–Civita connection ∇^{LC}.

Kähler condition

$$\mathrm{Hol(g)} \subseteq \mathrm{U}(n) \iff \nabla^{LC} J = 0 \iff \nabla^{LC} \omega = 0 \iff N_J = 0 \text{ and } d\omega = 0.$$

On complex manifold, a Hermitian metric is Kähler exactly when its fundamental $(1,1)$-form is closed (hence symplectic).

Equivalently, the Chern and Levi–Civita connections on the tangent bundle agree.

Local potentials and basic examples

If a real function ϕ has positive complex Hessian, then

$$\omega = i\partial\bar{\partial}\phi$$

is a Kähler form.

Examples

- Flat space: $\phi = \frac{1}{2}\sum |z_j|^2$.
- The unit disc: $\phi = -\frac{1}{2}\log(1 - |z|^2)$.
- A complex torus inherits a flat Kähler metric from $\mathbb{C}^n$.
- Riemann surfaces.

.

The Fubini–Study form

On $U_i \subset \mathbb{CP}^n$ with affine coordinates w, set

$$\omega_{FS} = \frac{i}{2\pi}\partial\bar{\partial}\log(1+|w|^2).$$

On overlaps, the potentials differ by the logarithm of the squared modulus of a nowhere-zero holomorphic function. Its $\partial\bar{\partial}$ vanishes. For a tangent vector u, positivity follows from

$$(1+|w|^2)|u|^2 - |\langle u, w\rangle|^2 > 0.$$

The form represents $c_1(\mathcal{O}(1))$, with the naive hermitian structure.

This is the symmetric metric on $\mathbb{CP}^n = \mathrm{U(n+1)/U(n)}$.

5

Kähler identities

The Kähler and Akizuki-Nakano identities

On a Kähler manifold (X, g, J, ω) we have the following relations between Dolbeault and the Lefschetz operators

$$\boxed{[L, \bar{\partial}^*] = -i\partial, \qquad [L, \partial^*] = i\bar{\partial}, \qquad [\Lambda, \bar{\partial}] = -i\partial^*, \qquad [\Lambda, \partial] = i\bar{\partial}^*.}$$

- These first-order identities are local.
- On a non-Kähler complex manifold, there are 0^{th}-order torsion corrections,
- (Akizuki–Nakano) On a hermitian bundle (E, h) with Chern connection D_E, we have

$$[\Lambda, \bar{\partial}_E] = -i(D_E^{1,0})^*$$

where D_e^* is the Chern connection on the dual bundle (E^*, h).

Dolbeaut Laplacians revisited – Bochner formulas

The Kähler identities and $\partial\bar\partial + \bar\partial\partial = 0$ imply

$$\Delta_{\nabla_E} = \nabla_E^* D_E + D_E D_E^* = \Delta_{\partial_E} + \Delta_{\bar\partial_E} \qquad \text{and} \qquad \Delta_{\partial_E} - \Delta_{\bar\partial_E} = -[iF_{D_E} \wedge \cdot\,.\Lambda] \,.$$

In the trivial case $E = \underline{\mathbb{C}}$, $D_E = d$, the curvature is 0, and so

$$\Delta_d = \Delta_\partial + \Delta_{\bar\partial} \qquad \text{and} \qquad \Delta_\partial = \Delta_{\bar\partial} \,.$$

De Rham harmonic forms split by bidegree, and each component is Dolbeault harmonic. On compact Kähler X,

$$H^k_{dR}(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}_{\bar\partial}(X), \qquad H^{p,q}_{\bar\partial}(X) \cong H^q(X, \Omega^p_X).$$

Hodge numbers and Hodge diamond

$$h^{p,q} = \dim_{\mathbb{C}} H^{p,q}_{\bar{\partial}}(X)$$

$$b_k = \sum_{p+q=k} h^{p,q}, \qquad h^{p,q} = h^{q,p} = h^{n-p,n-q}.$$

For projective space,

$$h^{p,q}(\mathbb{CP}^n) = \begin{cases} 1, & p = q, \\ 0, & p \neq q. \end{cases}$$

For a genus-g curve, the Hodge diamond is

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}.$$

Two obstructions to being Kähler

The Betti numbers of a Kähler manifold satisfy

$$b^{2k+1} = 0 \mod 2 \qquad \text{and} \qquad b^{2k} \geq 1 : .$$

- A Hopf manifold $S^{2n-1} \times S^1$, $n \geq 2$, has $b_1 = 1$, so it cannot be Kähler.

Holomorphic forms are de Rham harmonic.

- On the Heisenberg group, right-invariant holomorphic 1-forms

$$\eta_1 = da, \quad \eta_2 = dc, \quad \eta_3 = db - c\, da$$

descend to G/Γ, and $d\eta_3 = \eta_1 \wedge \eta_2 \neq 0$. Thus the Iwasawa manifold is not Kähler.