# Complex Geometry

## Meromorphic functions, bundles, divisors and Hermitian geometry

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1

# Meromorphic functions

# Fractions of holomorphic germs

The local ring $\mathcal{O}_{X,x}$ is a UFD (Lecture one), so we can consider the associated field of fractions to it:

$$\mathcal{K}_{X,x} = \mathrm{Frac}(\mathcal{O}_{X,x}).$$

**Definition**

A meromorphic function is a section of the sheaf with stalks $\mathcal{K}_{X,x}$.

A meromorphic function is **locally** a quotient $g/h$, with $h$ a non-zero holomorphic germ. Compatible local quotients form the sheaf $\mathcal{K}_X$.

On a connected complex manifold, $\mathcal{K}_X(X)$ is a field.

# Meromorphic functions on curves

Because $\mathcal{O}_{X,x}$ is a UFD, cancel common irreducible factors to obtain

$$f = \frac{g}{h}, \qquad (g, h) \text{ coprime as germs.}$$

On a Riemann surface $\Sigma$, a non-zero germ is $z^k u(z)$ with $u$ a unit.

$$\mathcal{K}(\Sigma) \cong \{f : \Sigma \to \mathbb{CP}^1 \cong \mathbb{C} \cup \{\infty\} \text{ holomorphic} : f \not\equiv \infty\}.$$

In higher dimensions coprime germs can vanish at the same point:

**Example**

$$z_1/z_2 \text{ on } \mathbb{C}^2$$

Zeros and poles can meet in codimension two.

# Algebraic independence and Siegel's theorem

Functions $f_1, \ldots, f_r$ are algebraically independent over $\mathbb{C}$ if

$$P(f_1, \ldots, f_r) \neq 0 \quad \text{for every non-zero } P \in \mathbb{C}[t_1, \ldots, t_r].$$

The transcendence degree is the largest size of an algebraically independent family.

It measures how many independent meromorphic parameters a manifold carries.

**Theorem**

If $X$ is compact and connected of complex dimension $n$, then

$$\operatorname{trdeg}_{\mathbb{C}} \mathcal{M}(X) \leq n.$$

I.e. any $n+1$ global meromorphic functions satisfy a non-trivial polynomial relation.

# A Schwarz estimate for high-order zeros

**Lemma**

If $f$ is holomorphic on a neighbourhood of $\overline{B_R}$ and vanishes to order at least $N$ at 0, then

$$\sup_{B_{R/2}} |f| \leq 2^{-N} \sup_{B_R} |f|.$$

**Proof**

Fix $0 \neq z \in D_\varepsilon(0)$ and consider the function

$$F := w \longmapsto w^{-k} f\left(w \frac{z}{|z|}\right),$$

which is holomorphic since $f$ has a zero of order $k$.

# Schwarz (cont'd) and Siegel

## Proof (cont'd).

On $\partial B_\varepsilon(0)$, we have $|F_z| \leq C\varepsilon^{-k}$.

By the maximum principle, the same bound holds for all $w \in B_\varepsilon(0)$.

Thus, taking $w = |z|$, we have

$$\left|F_z\left(|z|\right)\right| = |z|^{-k}|f(z)| \leq C\varepsilon^{-k}.$$

□

## Siegel's proof (Idea).

Use compactness to find suitable finite cover for $n+1$ meromorphic functions.

Construct a polynomial that "violates" Schwarz's growth, so it must vanish! □

# Algebraic dimension

**Definition**

For compact connected $X$, define the **algebraic dimension**

$$a(X) = \operatorname{trdeg}_{\mathbb{C}} \mathcal{K}_X(X), \qquad 0 \leq a(X) \leq \dim_{\mathbb{C}} X.$$

On $\mathbb{CP}^n$, the functions $z_1/z_0, \ldots, z_n/z_0$ are algebraically independent

Siegel gives the reverse bound. Hence

$$a(\mathbb{CP}^n) = n.$$

It is not hard to show that, in fact $\mathcal{K}_{\mathbb{CP}^n}(\mathbb{CP}^n) \cong \mathbb{C}(\xi_1, \ldots, \xi_n)$.

2

# Holomorphic vector bundles

# Local trivializations and transition functions

A rank-$r$ holomorphic bundle has local trivializations $E|_{U_i} \cong U_i \times \mathbb{C}^r$ with holomorphic transition functions

$$g_{ij} : U_i \cap U_j \to GL(r, \mathbb{C}), \qquad g_{ij} g_{jk} = g_{ik}.$$

Changing trivializations replaces $g_{ij}$ by $a_i g_{ij} a_j^{-1}$.

Thus rank-$r$ bundles correspond to the Čech cohomology **set**

$$\check{H}^1(X, GL(r, \mathcal{O}_X)).$$

For $r > 1$, this is generally a pointed set rather than a group.

# Bundle operations and sheaves

Direct sums, tensor products, duals, and exterior powers preserve holomorphicity.

$$E \oplus F \qquad E \otimes F \qquad E^* \qquad \Lambda^k E \qquad \dots$$

The holomorphic tangent bundle is $T^{1,0}X$, with transition matrices given by holomorphic Jacobians.

For a holomorphic bundle morphism of **constant rank**, the kernel, image, and cokernel are holomorphic bundles.

**Rank matters**

With varying rank these objects are sheaves, and need not be vector bundles.

# Holomorphic sections & the bundle Dolbeault operator

In a holomorphic frame $e_1, \dots, e_r$, define

$$\bar\partial_E\Big(\sum_j \alpha_j \otimes e_j\Big) = \sum_j \bar\partial \alpha_j \otimes e_j.$$

This is **independent** of the chosen frame, since we have holomorphic transition functions.

$\bar\partial_E$ extends naturally to $E$-valued forms:

$$\bar\partial_E(\alpha \wedge s) = \bar\partial\alpha \wedge s + (-1)^{\deg \alpha}\alpha \wedge \bar\partial_E s, \qquad \bar\partial_E^2 = 0.$$

**Theorem**

A smooth complex bundle with a $(0,1)$-operator satisfying the Leibniz rule and $\bar\partial_E^2 = 0$ admits a unique holomorphic structure inducing that operator.

# Bundle Dolbeault cohomology

Using the condition $\bar{\partial}_E^2 = 0$, we define

$$H^{p,q}_{\bar{\partial}_E}(X, E) = \frac{\ker(\bar{\partial}_E : \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q+1}(E))}{\mathrm{im}(\bar{\partial}_E : \mathcal{A}^{p,q-1}(E) \to \mathcal{A}^{p,q}(E))}.$$

Again, using $\bar{\partial}$-Poincaré lemma, we have the acyclic resolution

$$0 \to \Omega^p_X(E) \to \mathcal{A}^{p,0}_X(E) \to \mathcal{A}^{p,1}_X(E) \to \mathcal{A}^{p,2}_X(E) \to \dots$$

Hence,

$$H^{p,q}_{\bar{\partial}_E}(X, E) \cong H^q(X, \Omega^p_X \otimes \mathcal{E}),$$

where $\mathcal{E}$ is the sheaf of holomorphic sections of $E$.

3

# Line bundles and projective maps

# Classification of smooth line bundles

The smooth exponential sequence is

$$0 \to \underline{\mathbb{Z}} \to \mathcal{A}^0(\mathbb{C}) \xrightarrow{e^{2\pi i(\cdot)}} \mathcal{A}^0(\mathbb{C})^\times \to 0.$$

The middle additive sheaf is acyclic, so the long exact sequence yields

$$H^1\big(X, C_X^\infty(\mathbb{C})^\times\big) \xrightarrow[\cong]{\delta := c_1} H^2(X, \mathbb{Z}).$$

The map $c_1$ is a group morphism: tensor product adds first Chern classes, and dualization changes the sign.

Real line bundles are analogously classified by $w_1 \in H^1(X, \mathbb{Z}/2)$:

$$0 \to \mathcal{A}^0(\mathbb{R}) \xrightarrow{\exp} \mathcal{A}^0(\mathbb{R})^\times \to \underline{\mathbb{Z}/2\mathbb{Z}} \to 0\,.$$

# The Picard group

**Definition (Picard group)**

$$\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^\times).$$

The holomorphic exponential sequence gives

$$H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to \mathrm{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X).$$

- A smooth line bundle admits a holomorphic structure exactly when its integral Chern class maps to zero in $H^2(X, \mathcal{O}_X)$.
- If a smooth line bundle admits **a** holomorphic structure, it admits $\mathrm{Pic}^0$-many

$$\mathrm{Pic}^0(X) \coloneqq \ker c_1 \cong H^1(X, \mathcal{O}_X)/\operatorname{im} H^1(X, \mathbb{Z}).$$

# The tautological bundle

Over $\mathbb{CP}^n$, define

$$\mathcal{O}(-1) = \{(\ell, v) : v \in \ell\} \subset \mathbb{CP}^n \times \mathbb{C}^{n+1}.$$

Its fibre over $[z]$ is the line $\mathbb{C}z$.

$$\mathcal{O}(1) = \mathcal{O}(-1)^*, \qquad \mathcal{O}(k) = \mathcal{O}(1)^{\otimes k}, \quad k \in \mathbb{Z}.$$

**Theorem**

The class $c_1(\mathcal{O}(1))$ is the positive generator of $H^2(\mathbb{CP}^n, \mathbb{Z})$.

# The canonical bundle and Euler sequence

For a complex $n$-manifold,

$$K_X = \Lambda^n(T^{1,0}X)^* = \Omega^n_X.$$

The Euler sequence on projective space is

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O} & \to & \mathcal{O}(1)^{\oplus(n+1)} & \to & T^{1,0}\mathbb{CP}^n & \to & 0 \\ & & 1 & \mapsto & (z_0, \ldots, z_n) & & & & \\ & & & & s & \mapsto & \pi_* \left( \sum_i s_i \partial_i \right) & & \end{array}$$

Taking determinants gives

$$K_{\mathbb{CP}^n} \cong \mathcal{O}(-n-1).$$

# Linear systems and base loci

Let $V = \langle s_0, \dots, s_N \rangle \subset H^0(X, L)$.

$$B(V) = \{x : s_0(x) = \cdots = s_N(x) = 0\}.$$

On its complement define

$$\varphi_V : X \setminus B(V) \to \mathbb{P}(V^*), \qquad x \mapsto [s_0(x) : \cdots : s_N(x)].$$

Changing a local frame multiplies all coordinates by the same non-zero function.

A change of basis gives a projective linear transformation.

**Idea**

Find a line bundle $L$ with **enough** sections such that $B\big(H^0(X, L)\big) = \emptyset$ and $\phi : X \to \mathbb{P}\big(H^0(X, L)\big)$ is an embedding.

# Iitaka and Kodaira dimensions

The section ring and its degree-zero fraction field are

$$R(X,L) = \bigoplus_{m \geq 0} H^0(X, L^m), \qquad Q_0(X,L) = \{s/t : s,t \in H^0(X,L^m),\ t \neq 0\}.$$

For compact connected $X$,

$$\kappa(X,L) = \begin{cases} -\infty, & H^0(X,L^m) = 0 \text{ for every } m > 0, \\ \operatorname{trdeg}_{\mathbb{C}} Q_0(X,L), & \text{otherwise.} \end{cases}$$

Ratios of sections are meromorphic, so $\kappa(X,L) \leq a(X) \leq \dim X$.

For $L = K_X$, we call $\kappa(X) = \kappa(X, K_X)$ the Kodaira dimension of $X$.

# The dimension of $\mathcal{O}(k)$

$$H^0(\mathbb{CP}^n, \mathcal{O}(k)) = \begin{cases} \mathbb{C}[z_0, \ldots, z_n]_k, & k \geq 0, \\ 0, & k < 0. \end{cases}$$

**Proof.**

A section pulls back to a homogeneous holomorphic function of degree $k$ on $\mathbb{C}^{n+1} \setminus \{0\}$.

Hartogs extension and the growth estimate identify it with a homogeneous polynomial when $k \geq 0$. □

Consequently $h^0(\mathcal{O}(k)) = \binom{n+k}{n}$ for $k \geq 0$. Hence,

$$\kappa(\mathbb{CP}^n, \mathcal{O}(k)) = \begin{cases} n, & k > 0, \\ 0, & k = 0, \\ -\infty, & k < 0. \end{cases}$$

Thus $\kappa(\mathbb{CP}^n) = -\infty$.

4

# Divisors and adjunction

# Divisors and orders of vanishing

A divisor is a locally finite sum

$$D = \sum_Y m_Y [Y], \qquad m_Y \in \mathbb{Z},$$

over irreducible analytic hypersurfaces. It is effective if all $m_Y \geq 0$.

At a generic smooth point of $Y$, any non-zero meromorphic function $f$ can be written as $f = g^{\operatorname{ord}_Y f} u$ with $g \in \mathcal{O}_x$ reduced and $u$ a unit. $\operatorname{ord}_Y f$ is the order of $f$ at $Y$.

Orders are additive under multiplication.

$$\begin{aligned} \operatorname{div}(f) : \mathcal{K}_X(X) &\to \operatorname{Div}(X) \\ f &\mapsto \sum_Y \operatorname{ord}_Y(f)[Y]. \end{aligned}$$

# Cartier data and the line bundle $\mathcal{O}(D)$

On a smooth complex manifold, a divisor has local meromorphic equations $f_i$ with

$$f_i/f_j \in \mathcal{O}_X^\times(U_i \cap U_j).$$

$$H^0(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times) \cong \mathrm{Div(X)}$$

These ratios glue to define an element in $H^1(X, \mathcal{O}_X^\times) \implies$ hol. line bundle $\mathcal{O}(D)$.

- $\mathcal{O}(D + D') = \mathcal{O}(D) \otimes \mathcal{O}(D')$.
- For compact connected $X$,

$$1 \to \mathbb{C}^\times \to \mathcal{K}(X)^\times \to \mathrm{Div}(X) \to \mathrm{Pic}(X)$$

# Sections and effective divisors

A non-zero holomorphic section $s$ of a line bundle determines an effective zero divisor, with multiplicities:

$$L \cong \mathcal{O}(\mathrm{div}(s)).$$

Conversely, an effective divisor $D$ has a canonical section of $\mathcal{O}(D)$ vanishing along $D$. More generally, a line bundle comes from a divisor exactly when it admits a non-zero meromorphic section.

$$\mathrm{Cl}(X) \coloneqq \mathrm{Div}(X)/\langle f \,|\, f \text{ is meromorphic}\rangle$$

For an effective divisor $D$, multiplication by its local equation gives

$$0 \to \mathcal{O}(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0.$$

Here $\mathcal{O}(-D)$ is the ideal of functions vanishing with the multiplicities prescribed by $D$.

# Normal bundle and adjunction

For a smooth hypersurface $Y \subset X$,

$$N_{Y/X} \cong \mathcal{O}(Y)|_Y.$$

Taking determinants of the short exact sequence

$$0 \to T^{1,0}Y \to T^{1,0}X|_Y \to N_{Y/X} \to 0.$$

gives the celebrated adjunction formula

$$K_Y \cong (K_X \otimes \mathcal{O}(Y))|_Y.$$

**Example**

A smooth degree-$d$ hypersurface $Y \subset \mathbb{CP}^n$ satisfies $K_Y \cong \mathcal{O}(d-n-1)|_Y$.