Complex Geometry

Cohomology, sheaves, and meromorphic functions

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November 5, 2026

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Almost complex structures

Almost complex structures

Question:

When does a manifold M admit a compatible complex manifold structure?

The problem can be separated into two parts

$$\begin{array}{ccc} \text{algebro-topological} & + & \text{analytical} \\ \text{(principal bundle)} & & \text{(PDE)}\,. \end{array}$$

Definition

An almost complex structure on a smooth real manifold M^{2n} is a smooth bundle endomorphism $J : TM \to TM$ satisfying $J^2 = -\mathrm{Id}$.

It defines complex scalar multiplication on each tangent space:

$$(a + ib)v = av + bJv.$$

Topological constraints

An almost complex structure is equivalent to the choice of a reduction of the **frame group** from $GL(2n, \mathbb{R})$ to $GL(n, \mathbb{C})$.

Low dimensions

- A closed surface admits an almost complex structure iff it is orientable.
- A closed four-manifold admits one iff there exists $h \in H^2(M; \mathbb{Z})$ such that

$$w_1(M) = 0, \qquad h \equiv w_2(M) \pmod 2, \qquad \langle h^2, [M] \rangle = 2\chi(M) + 3\sigma(M).$$

- In dimension 6, there exist non-homological obstructions.

These are **topological** existence criteria; integrability is a further **differential** condition.

Type decomposition of the tangent bundle

Complexify TM and extend J complex linearly. Since $J^2 + 1 = (J - i)(J + i) = 0$ we have a **canonical** splitting

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M, \qquad T^{1,0} = \ker(J - i), \quad T^{0,1} = \ker(J + i).$$

Complex conjugation exchanges the two eigenbundles.

Dualizing and taking exterior powers gives the decomposition

$$\Lambda^k T^*M \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q} T^*M, \qquad \Lambda^{p,q} T^*M = \Lambda^p (T^{1,0}M)^* \otimes \Lambda^q (T^{0,1}M)^*.$$

We write $\mathcal{A}^{p,q}(M)$ for smooth sections of $\Lambda^{p,q} T^*M$.

The exterior differential on an almost complex manifold

The exterior differential $d : \mathcal{A}^k \to \mathcal{A}^{k+1}$ decomposes into local operators

$$d^{(p,q)} : \mathcal{A}^{p,q} \to \mathcal{A}^{p',q'}$$

$$d = \mu + \partial + \bar{\partial} + \bar{\mu}.$$

Operator	Bidegree
μ	$(2,-1)$
∂	$(1,0)$
$\bar{\partial}$	$(0,1)$
$\bar{\mu}$	$(-1,2)$

Each is a **graded derivation**.

The Nijenhuis obstruction

The operator μ is tensorial (function-linear) and can be identified with the **Nijenhuis** tensor:

$$N(X, Y) = \pi^{0,1}[X, Y]. \text{ with } X, Y \in \Gamma(T^{1,0}M).$$

Proof.

For $(0, 1)$-form η, X, Y as above, we get

$$(\mu\eta)(X, Y) = d\eta(X, Y) = X\eta(Y) - Y\eta(X) - \eta([X, Y]) = -\eta([X, Y]) = -\eta(N(X, Y)).$$

□

Interpretation: $N = 0$ exactly when $T^{1,0}M$ is involutive under the Lie bracket.

There exist other conventions for the Nijenhuis tensor. They only differ by constants.

Identities from $d^2 = 0$ and equivalent conditions

Separating d^2 by bidegree gives, we get

$$\mu^2 = 0, \qquad \mu\partial + \partial\mu = 0, \qquad \partial^2 + \mu\bar{\partial} + \bar{\partial}\mu = 0 \quad \text{and} \quad \mu\bar{\mu} + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\mu}\mu = 0,$$

together with their conjugates.

If $\mu = \bar{\mu} = 0$, the above relations reduce to

$$d = \partial + \bar{\partial}, \qquad \partial^2 = \bar{\partial}^2 = 0, \qquad \partial\bar{\partial} + \bar{\partial}\partial = 0.$$

Proposition

On an almost complex manifold, the following are equivalent:

1. $T^{1,0}M$ is involutive.
2. $\mu = \bar{\mu} = 0$.
3. $\bar{\partial}^2 = 0$ on all forms.

Newlander–Nirenberg and consequences

Theorem (Newlander-Nirenberg)

A smooth almost complex structure admits compatible local holomorphic coordinates if and only if its Nijenhuis tensor vanishes.

- Necessity follows in holomorphic coordinates: $T^{1,0}$ is spanned by commuting $\partial/\partial z_j$.
- Sufficiency is a **deep** analytic result.

Frobenius gives **smooth** coordinates. The hard part is to prove they are **analytic**.

- In real dimension two, the obstruction vanishes for dimension reasons. Every smooth almost complex surface is a Riemann surface.
- If $Z \subset X$ has J-invariant tangent spaces and X is complex, then the induced almost complex structure on Z is integrable.

2
Cohomology

The Dolbeault double complex

On a complex manifold, we have

$$d = \partial + \bar{\partial} \qquad \text{with} \qquad \partial : \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q} \quad \text{and} \quad \bar{\partial} : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}.$$

$$d^2 = 0 \qquad \Longrightarrow \qquad \partial^2 = \bar{\partial}^2 = 0 \quad \text{and} \quad \partial\bar{\partial} = -\bar{\partial}\partial.$$

Fixing p gives a (differential) complex

$$0 \longrightarrow \mathcal{A}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{A}^{p,n} \longrightarrow 0.$$

Its cohomology measures the failure of globally solving $\bar{\partial}\beta = \alpha$.

Dolbeault and de Rham cohomology

We have cohomology groups measuring the failure of the complex to be exact:

$$H^k_{\mathrm{dR}}(X;\mathbb{C}) = \frac{\ker(d : \mathcal{A}^k \to \mathcal{A}^{k+1})}{\mathrm{im}(d : \mathcal{A}^{k-1} \to \mathcal{A}^k)}.$$

$$H^{p,q}_{\bar\partial}(X) = \frac{\ker(\bar\partial : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1})}{\mathrm{im}(\bar\partial : \mathcal{A}^{p,q-1} \to \mathcal{A}^{p,q})}.$$

In particular, $H^{p,0}_{\bar\partial}(X)$ is the space of (global) holomorphic p-forms.

If X is closed, $H^{0,0}_{\bar\partial}(X) \cong H^k_{dR}(X,\mathbb{C}) \cong \mathbb{C}$.

Complex conjugation gives $H^{p,q}_{\partial} \cong \overline{H^{q,p}_{\bar\partial}}$.

Extraordinary cohomologies: Bott–Chern and Aeppli

Since $(\mathcal{A}^{p,q}, \partial, \bar{\partial})$ is a **double-complex**, we have the extraordinary cohomologies

$$H^{p,q}_{BC}(X) = \frac{\ker \partial \cap \ker \bar{\partial} \subset \mathcal{A}^{p,q}}{\mathrm{im}(\partial\bar{\partial} : \mathcal{A}^{p-1,q-1} \to \mathcal{A}^{p,q})},$$

$$H^{p,q}_{A}(X) = \frac{\ker(\partial\bar{\partial} : \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q+1})}{\partial\mathcal{A}^{p-1,q} + \bar{\partial}\mathcal{A}^{p,q-1}}.$$

The identity map on forms induces maps between the different cohomologies

$$H^{p,q}_{BC} \longrightarrow H^{p,q}_{\bar{\partial}} \longrightarrow H^{p,q}_{A}, \qquad H^{p,q}_{BC} \longrightarrow H^{p,q}_{\partial} \longrightarrow H^{p,q}_{A}.$$

$$H^{p,q}_{BC} \longrightarrow H^{p+q}_{\mathrm{dR}}(X;\mathbb{C}), \qquad H^{k}_{\mathrm{dR}}(X;\mathbb{C}) \longrightarrow \bigoplus_{p+q=k} H^{p,q}_{A}.$$

The one-variable $\bar{\partial}$ equation

Let f be smooth near a closed disc $\overline{B}$. For $z \in B$, put

$$Tf(z) = \frac{1}{2\pi i} \int_B \frac{f(w)}{w - z}\, dw \wedge d\bar{w}.$$

Then

$$\bar{\partial}(Tf) = f(z)\, d\bar{z},$$

by the Cauchy formula from Lecture 1

$$h(z) = \frac{1}{2\pi i} \int_{\partial B} \frac{h(w)}{w - z}\, dw + \frac{1}{2\pi i} \int_B \frac{h_{\bar{w}}(w)}{w - z}\, dw \wedge d\bar{w}.$$

The Cauchy transform: proof mechanism

Near a point z_0, choose a smooth cut-off $\chi = 1$ and split

$$f = \chi f + (1 - \chi)f = f_1 + f_2.$$

- The second term has support away from z_0, so its transform is holomorphic in a neighbourhood.
- Extend f_1 by zero to $\mathbb{C}$. Translation $w = z + \zeta$ moves derivatives onto the smooth numerator.
- The kernel $1/\zeta$ is locally integrable, so the transform is smooth.
- Cauchy–Green gives $\partial_{\bar{z}}(Tf_1) = f_1$.

Equivalently, $\partial_{\bar{z}}(1/(\pi z)) = \delta_0$ in the distributional sense.

The $\bar{\partial}$-Poincaré lemma

Lemma

Let $\alpha \in \mathcal{A}^{p,q}(U)$, $q > 0$, with $\bar{\partial}\alpha = 0$. On a sufficiently small polydisc $D \Subset U$, there exists $\beta \in \mathcal{A}^{p,q-1}(D)$ with

$$\alpha = \bar{\partial}\beta.$$

Proof (Idea).

use the one-variable Cauchy transform to remove terms containing $d\bar{z}_n$, then proceed inductively through the remaining coordinates.

The holomorphic factors dz_I act as coefficients, so the argument works for every p. □

Dolbeault cohomology of a polydisc

Dolbeault–Grothendieck lemma

For a polydisc $D \subset \mathbb{C}^n$,

$$H^{p,q}_{\bar\partial}(D) = \begin{cases} \Omega^p(D), & q = 0, \\ 0, & q > 0, \end{cases}$$

where $\Omega^p(D)$ denotes holomorphic p-forms.

Positive-degree closed forms have global primitives on the polydisc.

To pass from smaller closed polydiscs to D, use an exhaustion and compatible corrections to the primitives. The $q = 1$ step uses holomorphic approximation.

Local exactness and global obstructions

The local lemma gives an exact sequence of *sheaves*

$$0 \longrightarrow \Omega^p_X \longrightarrow \mathcal{A}^{p,0}_X \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1}_X \xrightarrow{\bar{\partial}} \cdots .$$

Local solvability need not produce a global solution: local primitives can disagree on overlaps.

The "best" way to keep track of these local-to-global obstructions is using sheaves.

3

Sheaves and stalks

Presheaves

A presheaf $\mathcal{F}$ of abelian groups assigns a group $\mathcal{F}(U)$ to every open set and restriction maps

$$r_{UV} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V), \qquad V \subset U.$$

They satisfy

$$r_{UU} = \mathrm{Id}, \qquad r_{VW} \circ r_{UV} = r_{UW}.$$

Examples: holomorphic functions, smooth differential forms, and sections of a bundle.

The sheaf axiom

Let $U = \bigcup_i U_i$ and $s_i \in \mathcal{F}(U_i)$.

Local compatibility and unique gluing

If $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, there is a unique $s \in \mathcal{F}(U)$ restricting to every s_i.

Equivalently,

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \longrightarrow \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is exact, where the last map takes differences of restrictions.

Stalks and germs

The stalk at x keeps only information arbitrarily close to x:

$$\mathcal{F}_x = \varinjlim_{x \in U} \mathcal{F}(U).$$

Two sections represent the same germ if they agree on a smaller neighbourhood of x. For the holomorphic structure sheaf,

$$\mathcal{O}_{X,x} \cong \mathbb{C}\{z_1, \ldots, z_n\}.$$

A section of a sheaf is determined by all its germs.

Sheafification

The étale space is the disjoint union

$$\mathrm{Et}(\mathcal{F}) = \coprod_{x \in X} \mathcal{F}_x \xrightarrow{p} X,$$

equipped with the topology generated by $\{s_x : x \in U\}$ for local sections s.

The sheafification $\mathcal{F}^+$ consists of continuous local sections of p.

- $\mathcal{F} \to \mathcal{F}^+$ induces isomorphisms on stalks.
- Every morphism from $\mathcal{F}$ to a sheaf factors uniquely through $\mathcal{F}^+$.

Example: Constant and locally constant data

The constant presheaf with value A assigns A to non-empty open sets, with identity restrictions. $\rightarrow$ It is not a sheaf, since local information cannot be glued!

Constant sheaf

The sheafification is $\underline{A}$, the sheaf of locally constant A-valued functions.

Check: for two disjoint connected open sets U, V, $\underline{A}(U \cup V) = A \times A$.

Kernels, images, and cokernels

For a sheaf morphism $\varphi : \mathcal{F} \to \mathcal{G}$,

- $U \mapsto \ker \varphi_U$ is already a sheaf.
- The image presheaf can fail to be a sheaf.
- Define $\operatorname{im} \varphi$ and $\operatorname{coker} \varphi$ by sheafifying the corresponding presheaves.

Exactness is local

A sequence of sheaves is exact if and only if its sequence of stalks is exact at every point.

The exponential sequence

On a complex manifold,

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}_X \xrightarrow{f \mapsto e^{2\pi i f}} \mathcal{O}_X^{\times} \longrightarrow 1$$

is exact as a sequence of sheaves of abelian groups.

Every nowhere-zero holomorphic function has local logarithms. **BUT**

z on $\mathbb{C}^{\times}$ has no global logarithm.

Surjectivity on stalks does not imply surjectivity on global sections.

Exact sequences in geometry

- For a closed complex submanifold $i : Y \hookrightarrow X$,

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to i_* \mathcal{O}_Y \to 0.$$

- For $x \in X$, the skyscraper sheaf $\mathbb{C}_x$ gives

$$0 \to \mathcal{I}_x \to \mathcal{O}_X \to \mathbb{C}_x \to 0.$$

- The local Poincaré lemmas give resolutions

$$0 \to \underline{\mathbb{C}} \to \mathcal{A}_X^0 \xrightarrow{d} \mathcal{A}_X^1 \to \cdots,$$

$$0 \to \Omega_X^p \to \mathcal{A}_X^{p,0} \xrightarrow{\bar{\partial}} \mathcal{A}_X^{p,1} \to \cdots.$$

Direct and inverse images

For a continuous $f : X \to Y$,

$$(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}V).$$

The inverse image $f^{-1}\mathcal{G}$ is the sheafification of

$$U \longmapsto \varinjlim_{f(U)\subset V} \mathcal{G}(V).$$

There is an adjunction

$$\mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}).$$

Inverse image is exact. Direct image is generally only left exact.

Ringed spaces and module sheaves

A ringed space is $(X, \mathcal{R})$ with a sheaf of rings $\mathcal{R}$. An $\mathcal{R}$-module sheaf has compatible module structures on its sections.

- Smooth manifolds carry C^∞_X.
- Complex manifolds carry $\mathcal{O}_X$.
- Holomorphic vector bundles give locally free $\mathcal{O}_X$-modules.
- Ideals and quotients describe subspaces and their equations.

4
Sheaf cohomology

Global sections and lifting obstructions

For a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0,$$

global sections give only

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G}) \to \Gamma(X, \mathcal{H}).$$

A section of $\mathcal{H}$ lifts locally. Differences of local lifts lie in $\mathcal{F}$ and may obstruct a global lift. Cohomology records this obstruction.

Injective resolutions and long exact sequences

A sheaf $\mathcal{I}$ is **injective** if maps into it extend across sheaf injections.

Every sheaf admits an injective resolution

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots .$$

Definition

$$H^q(X, \mathcal{F}) = H^q\big(\Gamma(X, \mathcal{I}^\bullet)\big).$$

Comparison maps between resolutions make this independent of the chosen resolution.

In degree zero, $H^0(X, \mathcal{F}) = \mathcal{F}(X)$.

The long exact sequence

Theorem

Every short exact sequence of sheaves

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

gives an induced long exact sequence

$$0 \to H^0(\mathcal{F}) \to H^0(\mathcal{G}) \to H^0(\mathcal{H}) \xrightarrow{\delta} H^1(\mathcal{F}) \to H^1(\mathcal{G}) \to H^1(\mathcal{H}) \xrightarrow{\delta} H^2(\mathcal{F}) \to \cdots .$$

The connecting map $s \mapsto \delta(s)$ vanishes exactly when s lifts globally.

This turns a geometric **existence** problem into a **vanishing** problem.

Showing that $\delta = 0$ is generally hard...showing $H^i(\mathcal{F}) = 0$ is **easier**

Flabby, fine, and acyclic sheaves

- Flabby sheaves have surjective restrictions $\mathcal{F}(U) \to \mathcal{F}(V)$ for $V \subset U$.
- Injective sheaves are flabby, and flabby sheaves are acyclic.
- On a smooth paracompact manifold, sheaves of C^∞_X-modules are fine and acyclic, by partitions of unity.

An acyclic resolution can replace an injective one:

$$0 \to \mathcal{F} \to \mathcal{B}^\bullet \quad \Longrightarrow \quad H^q(X, \mathcal{F}) = H^q\big(\Gamma(X, \mathcal{B}^\bullet)\big).$$

de Rham and Dolbeault as sheaf cohomology

Theorem

$$H^k(X, \underline{\mathbb{C}}) \cong H^k_{\mathrm{dR}}(X; \mathbb{C}).$$
$$H^q(X, \Omega^p_X) \cong H^{p,q}_{\bar\partial}(X).$$

Proof.

Proof (Idea) We have **acyclic** resolutions

$$0 \to \underline{\mathbb{C}} \to \mathcal{A}^0_X \xrightarrow{d} \mathcal{A}^1_X \xrightarrow{d} \mathcal{A}^2_X \xrightarrow{d} \dots$$

$$0 \to \Omega^p_X \to \mathcal{A}^{p,0}_X \xrightarrow{\bar\partial} \mathcal{A}^{p,1}_X \xrightarrow{\bar\partial} \mathcal{A}^{p,2}_X \xrightarrow{\bar\partial} \dots$$

The local Poincaré lemma supplies exactness. Partitions of unity supply acyclicity of the smooth-form sheaves. Global cohomology survives because local primitives need not glue. □

5

Čech cohomology

Cochains on an open cover

For a cover $\mathcal{U} = \{U_i\}$, we define the Čech complex $C^q(\mathcal{U}, \mathcal{F})$ and its cohomology $\check{H}^q(\mathcal{U}, \mathcal{F})$.

$$C^q(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \cdots < i_q} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_q}).$$

$$(\delta s)_{i_0 \cdots i_{q+1}} = \sum_{k=0}^{q+1} (-1)^k s_{i_0 \cdots \widehat{i_k} \cdots i_{q+1}} \big|_{U_{i_0 \cdots i_{q+1}}}.$$

Deleting two indices in either order gives cancelling terms, so $\delta^2 = 0$.

- A 0-cocycle is a family of sections agreeing on overlaps. Thus $\check{H}^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$.
- A 1-cocycle satisfies $s_{ij} + s_{jk} = s_{ik}$ on triple overlaps.
- A 1-coboundary has $s_{ij} = t_j - t_i$.

Refinement and Leray covers

The groups $\check{H}^q(\mathcal{U}, \mathcal{F})$ may depend on the cover.

$$\check{H}^q(X, \mathcal{F}) = \varinjlim_{\mathcal{U}} \check{H}^q(\mathcal{U}, \mathcal{F}).$$

Leray's cover theorem

For **good** covers, we gave

$$\check{H}^q(\mathcal{U}, \mathcal{F}) \cong H^q(X, \mathcal{F}).$$

On manifolds, the direct-limit Čech cohomology agrees with sheaf cohomology.

A computation on the projective line

Cover $\mathbb{CP}^1$ by $U_0 = \mathbb{C}_z$ and $U_1 = \mathbb{C}_{1/z}$. Their overlap is $\mathbb{C}^\times$. A holomorphic cocycle f on $\mathbb{C}^\times$ has a Laurent expansion

$$f(z) = \sum_{k \geq 0} a_k z^k + \sum_{k < 0} a_k z^k.$$

The first sum is entire in z; the second is entire in $1/z$ and vanishes at infinity. Thus f is a difference of sections on the two charts, giving

$$H^0(\mathbb{CP}^1, \mathcal{O}) = \mathbb{C} \qquad \text{and} \qquad H^1(\mathbb{CP}^1, \mathcal{O}) = 0.$$

The two-chart cover is acyclic for $\mathcal{O}$.