

Complex Geometry

Holomorphic functions and complex manifolds

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Lecture notes available at <https://enric-sf.github.io/>

Course overview

The course aims to develop the main ideas of classical complex geometry and some of the field's most important results of the twentieth century:

- The Kodaira embedding theorem
- Kodaira–Spencer deformation theory and the Tian–Todorov lemma
- Yau's proof of the Calabi–Yau conjecture

Required background

- Complex analysis
- Differential geometry

Desirable background

- Riemannian geometry
- Vector bundles and characteristic classes

References

- D. Huybrechts *Complex Geometry: An Introduction*
- A. Moroianu *Kähler Geometry*
- J.-P. Demailly *Complex Analytic and Differential Geometry*
- S. Kobayashi *Differential Geometry of Complex Vector Bundles*
- K. Kodaira *Complex Manifolds and Deformation of Complex Structures*
- R. O. Wells *Differential Analysis on Complex Manifolds*

Lecture notes and contact

Lecture notes

Download the course notes

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Corrections and questions

Please report mistakes or typos by email:

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1

Holomorphic maps

Complex differentiability

Let $U \subset \mathbb{C}^n$ be open and $f : U \rightarrow \mathbb{C}^m$ real differentiable.

Holomorphicity

f is holomorphic if its real differential is complex linear at every point:

$$df_z \circ J_n = J_m \circ df_z.$$

Here J_n is the linear map induced by multiplication by i on $\mathbb{R}^{2n} \cong \mathbb{C}^n$.

In the basis $\{e_1, \dots, e_{2n}\}$ satisfying $i \cdot e_{2k+1} = e_{2k+2}$, we have

$$J = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & 0 & -1 & \\ & & 1 & 0 & \\ & & & & \dots \end{pmatrix}$$

Wirtinger derivatives

For $z_j = x_j + iy_j$, set

$$\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j}), \quad \partial_{\bar{z}_j} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j}).$$

$$df = \sum_j \frac{\partial f}{\partial z_j} dz_j + \sum_j \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j.$$

Cauchy–Riemann equations

f is holomorphic exactly when $\partial f / \partial \bar{z}_j = 0$ for every j .

For $f = u + iv$: $u_{x_j} = v_{y_j}$ and $u_{y_j} = -v_{x_j}$. = Examples:

$$f(z) = z^2 : \quad f_z = 2z, \quad f_{\bar{z}} = 0; \quad g(z) = \bar{z} : \quad g_z = 0, \quad g_{\bar{z}} = 1.$$

Complex and real Jacobians

We can consider the complexified derivative

$$Df(z_0)^{\mathbb{C}} : T_{z_0}\mathbb{R}^{2n} \otimes \mathbb{C} \longrightarrow T_{f(z_0)}\mathbb{R}^2 \otimes \mathbb{C}.$$

$$T_{z_0}\mathbb{R}^{2n} \otimes \mathbb{C} \cong \langle \partial_{z_j}, \partial_{\bar{z}_j} \rangle_{j=1, \dots, n} \qquad T_{f(z_0)}\mathbb{R}^2 \otimes \mathbb{C} \cong \langle \partial_\omega, \partial_{\bar{\omega}} \rangle$$

For $f : U \subset \mathbb{C}^n \rightarrow \mathbb{C}^n$ holomorphic, the complexified Jacobian takes the form

$$Df = \begin{pmatrix} \frac{\partial f}{\partial z_i} & 0 \\ 0 & \frac{\partial f}{\partial \bar{z}_i} \end{pmatrix},$$

In particular,

$$\det_{\mathbb{R}}(df) = \left| \det_{\mathbb{C}} \left(\frac{\partial f_i}{\partial z_j} \right) \right|^2 \geq 0.$$

Holomorphic inverse and implicit functions

Inverse function theorem

If $f : U \subset \mathbb{C}^n \rightarrow \mathbb{C}^n$ is holomorphic and $\det_{\mathbb{C}} df_{z_0} \neq 0$, then f has a holomorphic local inverse near z_0 .

Implicit function theorem

Write $(w, z) \in \mathbb{C}^{n-m} \times \mathbb{C}^m$. If $F(w_0, z_0) = c$ and $\det(\partial F / \partial z) \neq 0$, then locally

$$F^{-1}(c) = \{(w, g(w))\}, \quad g \text{ holomorphic.}$$

Proof (Idea).

The smooth (real) theorem gives a smooth inverse; differentiate the inverse identity to obtain complex linearity. □

Regular level sets and local normal forms

If df has constant complex rank r near a point, suitable holomorphic coordinates give

$$f(z_1, \dots, z_n) = (z_1, \dots, z_r, 0, \dots, 0).$$

- A submersion has local holomorphic sections.
- An immersion has a local holomorphic left inverse onto its source.
- A regular level set of $F : \mathbb{C}^n \rightarrow \mathbb{C}^m$ has complex dimension $n - m$.

Example: $Z(z_1 z_2 - 1) \subset \mathbb{C}^2$ is a smooth complex curve, parametrized by $t \mapsto (t, t^{-1})$.

One is tempted to define complex (sub)manifold as the preimage of a regular value of a holomorphic value.

Cauchy Integral Formula

Theorem (Cauchy Integral Formula)

Let $K \subseteq \mathbb{C}$ be a compact subset with piecewise C^1 boundary $C = \partial K$, and $f : K \rightarrow \mathbb{C}$ a differentiable function. Then for $z \in K \setminus \partial K$, we have

$$2\pi i f(z) = \int_{\partial K} \frac{f(w, \bar{w})}{w - z} dw + \int_K \frac{\partial f}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w - z} \quad (1)$$

Proof (Idea)

Set $z = 0$ and take $\delta > 0$. Since $\frac{f}{w} \in L^1(K)$, we have

$$\int_{K \setminus B_\delta(0)} d \left(\frac{f(w, \bar{w})}{w} \right) dw = - \int_{K \setminus B_\delta(0)} \frac{\partial f}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w} .$$

Proof (Cont'd).

By Stokes' theorem

$$\begin{aligned}\int_{K \setminus B_\delta(0)} d\left(\frac{f(w, \bar{w})}{w}\right) dw &= \int_{\partial K} \frac{f(w, \bar{w})}{w} dw - \int_{\partial B_\delta} \frac{f(w, \bar{w})}{w} dw \\ &= \int_{\partial K} \frac{f(w, \bar{w})}{w} dw - \int_0^{2\pi} f(\delta, \theta) i d\theta .\end{aligned}$$

Taking $\delta \rightarrow 0$, the claim follows. □

If f is holomorphic, we recover the "usual" CIF:

$$f(z) = \frac{1}{2\pi i} \int_{\partial K} \frac{f(w)}{w - z} dw .$$

Cauchy formula on a polydisc

Let $D_R(a) = \prod_{j=1}^n \{|z_j - a_j| < R_j\}$ and

$$T_R(a) = \prod_{j=1}^n \{|\zeta_j - a_j| = R_j\}.$$

For f holomorphic near $\overline{D_R(a)}$,

$$f(z) = \frac{1}{(2\pi i)^n} \int_{T_R(a)} \frac{f(\zeta) d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)}.$$

Apply the one-variable Cauchy formula successively in each coordinate (**Fubini's theorem**).

Power series and Cauchy estimates

Expand each Cauchy kernel as a geometric series $\frac{1}{w-z} = \sum_{k \geq 0} \frac{z^k}{w^{k+1}}$. Then

$$f(z) = \frac{1}{2\pi i} \int_C \sum_{k \geq 0} z^k \frac{f(w)}{w^{k+1}} dw = \frac{1}{2\pi i} \sum_{k \geq 0} z^k \int_C \frac{f(w)}{w^{k+1}} dw = \sum_{\alpha \in \mathbb{N}^n} \frac{\partial^\alpha f(a)}{\alpha!} (z-a)^\alpha,$$

by Lebesgue dominated convergence and uniqueness of Taylor expansions.

Holomorphic functions are smooth and analytic on their domain.

$$|\partial^\alpha f(a)| \leq \frac{\alpha!}{R^\alpha} \sup_{T_R(a)} |f| \quad \rightsquigarrow \quad |f| \leq C(1+|z|)^D \iff f \text{ polynomial}.$$

Here $\alpha! = \prod_j \alpha_j!$ and $R^\alpha = \prod_j R_j^{\alpha_j}$.

Identity principle and maximum principle

On a connected open set $U \subset \mathbb{C}^n$:

- If two holomorphic functions agree on a nonempty open subset, they agree on U .
- A nonconstant scalar holomorphic function is an open map.
- If $|f|$ has an interior local maximum, f is constant.

Identity principle: proof idea

The set where all derivatives of $f - g$ vanish is closed and, by the power series expansion, open.

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Hartogs and Weierstrass Preparation Theorem

Hartogs extension

Hartogs phenomenon, $n \geq 2$

If $0 < r_j < R_j$ and f is holomorphic on

$$D_R \setminus \overline{D_r},$$

then f extends uniquely to a holomorphic function on D_R .

$1/z$ on $D_1(0) \setminus \{0\}$ is a counterexample for $n = 1$. In several variables, the *extra* coordinate supplies an open region where a Cauchy integral agrees with f

Corollary

An isolated singularity of a holomorphic function on $\mathbb{C}^n$, $n \geq 2$, is always **removable**.

Hartogs extension: the Cauchy construction

Write $z = (z_1, w)$ and choose $r_1 < \rho < R_1$. Define

$$F(z_1, w) = \frac{1}{2\pi i} \int_{|\xi|=\rho} \frac{f(\xi, w)}{\xi - z_1} d\xi, \quad |z_1| < \rho.$$

- The contour stays outside the inner polydisc for every w .
- Parameter integration makes F holomorphic.
- When w lies outside the inner w -polydisc, Cauchy's formula gives $F = f$.
- Varying ρ defines F for every $(z_1, w) \in D_R$.
- Uniqueness follows from the identity principle

Boundary integration is holomorphic

Lemma

Consider the open cylinder $U \times V$ with $U \subseteq \mathbb{C}^n$ open, and $V \subseteq \mathbb{C}$ a neighbourhood of $\partial B_\epsilon(z_0)$ and let $f : V \times U \rightarrow \mathbb{C}$ a holomorphic function. Then

$$g(z_1, \dots, z_n) := \int_{\partial B_\epsilon(z_0)} f(\xi, z_1, \dots, z_n) d\xi$$

is a holomorphic function on U .

Proof.

If f were holomorphic



Weierstrass polynomials

Write $(z, w) \in \mathbb{C} \times \mathbb{C}^{n-1}$. A Weierstrass polynomial of degree d is

$$P(z, w) = z^d + a_1(w)z^{d-1} + \cdots + a_d(w), \quad a_j(0) = 0,$$

with holomorphic coefficients $a_j(w) : \mathbb{C}^{n-1} \rightarrow \mathbb{C}$.

Example

$P(z, w) = z^2 - w$ has roots $\pm\sqrt{w}$. The roots need not vary holomorphically, although their symmetric functions do.

This distinction is the key to the preparation theorem.

Weierstrass preparation

Weierstrass Preparation theorem

Let $f : D_\epsilon(0) \rightarrow \mathbb{C}$ with $f(0,0) = 0$ and $f(z_1, 0, \dots, 0) \not\equiv 0$. Then for some smaller ball $D_{\epsilon'}(0)$ there exists a unique decomposition:

$$f = g \cdot h$$

where g is a Weierstrass polynomial in z_1 , and $h : D_{\epsilon'}(0) \rightarrow \mathbb{C}$ is a holomorphic function without zeros.

The theorem is more naturally phrased in terms of germs, which we will introduce next.

Proof sketch of WPT

For small w , the zeros of $f_w(z) = f(z, w)$ are given by $a_1(w), \dots, a_d(w)$. Define:

$$g(z, w) = \prod_{j=1}^d (z_1 - a_j(w)), \quad h = \frac{f}{g}$$

- We need to show that g and h are holomorphic in z_1 and w .
- Showing holomorphicity in w for g amounts to showing the symmetric polynomials $s_k(w) = \sum_j a_j^k(w)$ are holomorphic. The argument principle counts d zeros, with multiplicity.
- By Cauchy Integral Formula, we have

$$s_k(w) = \frac{1}{2\pi i} \int_{|\xi|=\rho} \xi^k \partial_\xi \log(f(\xi, w)) d\xi.$$

- The integrand is holomorphic in a neighbourhood of the contour, so we may differentiate under the integral sign.
- Similarly for h .

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Germs and analytic sets

The ring of holomorphic germs

Let $U, V \subseteq \mathbb{C}^n$ open sets containing the origin, $f : U \rightarrow \mathbb{C}$ and $g : V \rightarrow \mathbb{C}$ holomorphic

We say the pairs (U, f) and (V, g) are **equivalent** if there exists an open $W \subseteq U \cap V$ such that $f|_W = g|_W$.

$\implies$ This is an **equivalence relation**.

Definition

An equivalence class $[(U, f)]$ is called a **holomorphic germ** at 0. The collection of holomorphic germs at 0 is the **holomorphic stalk** at 0, denoted $\mathcal{O}_n$

Theorem

$$\mathcal{O}_n \cong \mathbb{C}\{z_1, \dots, z_n\},$$

where the braces denote convergent power series.

Properties of $\mathcal{O}_n$

Structure theorem

The holomorphic stalk $\mathcal{O}_n$ is

- a local ring,
- a unique factorisation domain (by WPT), and
- a Noetherian ring (by Weierstrass Division theorem).

The stalk of smooth germs $\mathcal{A}_n$ is a local ring, it is **not** a UFD or a Noetherian ring.

The root of all smooth "evil": $f(x) = \exp\left(\frac{-1}{x^2}\right)$

Analytic sets and their germs

A subset $Z \subset U$ is analytic if locally

$$Z = Z(f_1, \dots, f_k) = \{f_1 = \dots = f_k = 0\}$$

for holomorphic functions f_i . Analytic germs are defined analogously

Analytic to holomorphic correspondence

$$Z \rightsquigarrow I(Z) := \{f \in \mathcal{O}_n : f|_Z = 0\}, \quad I \rightsquigarrow Z(I).$$

- $Z(I + J) = Z(I) \cap Z(J)$.
- $Z(IJ) = Z(I) \cup Z(J)$.
- Both $I(\cdot)$ and $Z(\cdot)$ reverse inclusions.

The local analytic Nullstellensatz

$$Z = Z(I(Z)) \quad I \subseteq I(Z(I)) .$$

Example: $(z_1) \neq (z_1^2)$ but $I(Z(z_1^2)) = (z_1)$.

Nullstellensatz theorem

$$I(Z(I)) = \sqrt{I} = \{f : f^k \in I \text{ for some } k \geq 1\}.$$

An analytic germ Z is irreducible if $Z = Z_1 \cup Z_2$ forces $Z = Z_1$ or $Z = Z_2$.

Analytic germs $\iff$ **radical** ideals

Irreducible germs $\iff$ **prime** ideals

4

Complex manifolds

Complex atlases

A complex n -manifold is a Hausdorff, second-countable space with charts

$$\varphi_i : U_i \longrightarrow V_i \subset \mathbb{C}^n$$

whose transition maps $\varphi_j \circ \varphi_i^{-1}$ are **biholomorphic**.

A compatible atlas extends uniquely to a maximal atlas.

- The underlying manifold is smooth of real dimension $2n$.
- Complex linear Jacobians give a canonical orientation. $(\mathrm{GL}(n, \mathbb{C}) \subseteq \mathrm{GL}(2n, \mathbb{R})^+$
- Coordinate changes must satisfy **analytic differential equations**.

Holomorphic maps and submanifolds

$f : X \rightarrow Y$ is holomorphic when every coordinate expression

$$\psi \circ f \circ \varphi^{-1}$$

is holomorphic.

A complex submanifold $Z^k \subset X^n$ has local equations

$$z_{k+1} = \cdots = z_n = 0.$$

If $f : X^n \rightarrow Y^m$ is holomorphic and y is a regular value, then

$f^{-1}(y)$ is a complex submanifold of dimension $n - m$.

as **expected!**

Compactness and holomorphic functions

Maximum principle on a compact manifold

Every holomorphic function on a connected compact complex manifold is constant.

$|f|$ attains a maximum, so f is locally and then globally constant.

Corollary

Compact complex submanifolds of $\mathbb{C}^N$ are finite sets of points.

There is no analogue of **Whitney embedding theorem** for compact manifolds!

Idea

Replace $\mathbb{C}^n$ by another "nice" space $\rightsquigarrow$ Kodaira Embedding theorem.

Examples (I): Complex projective space

$$\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^\times.$$

Homogeneous coordinates satisfy $[z_0 : \cdots : z_n] = [\lambda z_0 : \cdots : \lambda z_n]$.

Charts:

$$U_i = \{z_i \neq 0\}, \quad \varphi_i([z]) = (z_0/z_i, \dots, \widehat{z_i/z_i}, \dots, z_n/z_i).$$

Transitions are holomorphic rational functions on their domains.

The quotient map $S^{2n+1} \rightarrow \mathbb{CP}^n$ is surjective, so $\mathbb{CP}^n$ is compact.

For $n = 1$, $\mathbb{CP}^1$ is the Riemann sphere $\mathbb{C} \cup \{\infty\} \cong S^2$.

Projective hypersurfaces and complete intersections

The vanishing locus of homogeneous polynomial p defines an analytic subset of $\mathbb{CP}^n$.

If $dp \neq 0$ on $Z(p) \setminus \{0\}$, then $X: (Z(p) \setminus \{0\})/\mathbb{C}^*$ is a compact complex manifold of dimension $n - 1$.

More generally, homogeneous $p_1, \dots, p_k$ with linearly independent differentials on their zero locus define a smooth complete intersection of dimension $n - k$.

Examples

- Fermat hypersurface : $z_0^d + z_1^d + \dots + z_n^d = 0$ in $\mathbb{CP}^n = n$.
- The (1, 1)-Segre embedding

$$\begin{aligned}\mathbb{CP}^1 \times \mathbb{CP}^1 &\rightarrow \mathbb{CP}^3 \\ ([s : t], [u : v]) &\longmapsto [su : sv : tu : tv]\end{aligned}$$

The image satisfies $x_0x_3 - x_1x_2 = 0 \rightsquigarrow$ Quadric surface in $\mathbb{CP}^3$.

Examples (II): Quotients

Theorem/Fact

A free, properly discontinuous action by a discrete group of biholomorphisms gives a complex manifold quotient.

Examples

- Flat tori: full lattice $\Lambda \subset \mathbb{C}^n$, $\rightsquigarrow \mathbb{T}^n: \mathbb{C}^n/\Lambda$.
- Hopf manifold: $\rho_\alpha(z) = \alpha z$, $\rightsquigarrow H_\alpha: (\mathbb{C}^n \setminus \{0\})/\langle \rho_\alpha \rangle \cong S^1 \times S^{2n-1}$.
- Iwasawa manifold: $G = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \subseteq \mathrm{GL}(3, \mathbb{C})$, $\Gamma = \{g \in G \mid a, b, c \in \mathbb{Z}[i]\}$.
 $\mathbb{I} := G/\Gamma \cong \mathbb{T}^1$ -fibration over $\mathbb{T}^2$.

Examples (III) (or lack thereof)

A **complex** Lie group is a Lie group whose multiplication is holomorphic.

Examples

$GL(n, \mathbb{C})$, $SL(n, \mathbb{C})$, $Sp(2n, \mathbb{C})$, ... **BUT NOT** $U(n)$ or $Sp(n)$!!

Theorem

Let G be a connected compact complex Lie group. Then $G \cong \mathbb{T}^n$ for some n .

Proof (abelian).

- Let $\mathfrak{g} = \text{Lie}(G)$ and $\text{Ad}_g : G \rightarrow GL(\mathfrak{g})$ for $g \in G$.
- Ad_g holomorphic + G compact $\implies \text{Ad}_g \equiv \text{Id}$.
- $\mathfrak{g}$ has trivial bracket $\implies G$ abelian.



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Almost complex structures

Almost complex structures

Definition

An almost complex structure on a smooth real manifold M^{2n} is a smooth bundle endomorphism $J : TM \rightarrow TM$ satisfying $J^2 = -\text{Id}$.

It defines complex scalar multiplication on each tangent space:

$$(a + ib)v = av + bJv.$$

Complex charts induce such a J , since holomorphic transition differentials commute with multiplication by i .

The converse requires an **integrability** condition.

Pseudo-holomorphic maps

For almost complex manifolds (M, I) and (N, J) , a smooth map $f : M \rightarrow N$ is pseudo-holomorphic when

$$df \circ I = J \circ df.$$

For $N = \mathbb{C}$, this becomes $df \circ I = i df$.

- This is a first-order differential equation.
- For integrable structures, it recovers holomorphic maps.
- An arbitrary almost complex structure may have very few local complex-valued holomorphic functions.

Topological constraints

An almost complex structure is equivalent to the choice of a reduction of the **frame group** from $GL(2n, \mathbb{R})$ to $GL(n, \mathbb{C})$.

Closed manifolds in low dimensions

- A closed surface admits an almost complex structure iff it is orientable.
- A closed four-manifold admits one iff there exists $h \in H^2(M; \mathbb{Z})$ such that

$$w_1(M) = 0, \quad h \equiv w_2(M) \pmod{2}, \quad \langle h^2, [M] \rangle = 2\chi(M) + 3\sigma(M).$$

These are **topological** existence criteria; integrability is a further **differential** condition.

Type decomposition of the tangent bundle

Complexify TM and extend J complex linearly. Since $J^2 + 1 = (J - i)(J + i) = 0$ we have a **canonical** splitting

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M, \quad T^{1,0} = \ker(J - i), \quad T^{0,1} = \ker(J + i).$$

Complex conjugation exchanges the two eigenbundles.

Dualizing and taking exterior powers gives the decomposition

$$\Lambda^k T^*M \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q} T^*M, \quad \Lambda^{p,q} T^*M = \Lambda^p(T^{1,0}M)^* \otimes \Lambda^q(T^{0,1}M)^*.$$

The exterior differential on an almost complex manifold

We write $\mathcal{A}^{p,q}(M)$ for smooth sections of $\Lambda^{p,q} T^*M$.

The exterior differential $d : \mathcal{A}^k \rightarrow \mathcal{A}^{k+1}$ decomposes into local operators $d^{(p,q)} : \mathcal{A}^{p,q} \rightarrow \mathcal{A}^{p',q'}$:

$$d = \mu + \partial + \bar{\partial} + \bar{\mu}.$$

Operator	Bidegree	Differential order
μ	$(2, -1)$	0
∂	$(1, 0)$	1
$\bar{\partial}$	$(0, 1)$	1
$\bar{\mu}$	$(-1, 2)$	0

Each is a **graded derivation**.

The Nijenhuis obstruction

The operator μ is tensorial (function-linear) and can be identified with the **Nijenhuis** tensor:

$$N(X, Y) = \pi^{0,1}[X, Y]. \text{ with } X, Y \in \Gamma(T^{1,0}M).$$

Proof.

For $(0, 1)$ -form η , X, Y as above, we get

$$(\mu\eta)(X, Y) = d\eta(X, Y) = X\eta(Y) - Y\eta(X) - \eta([X, Y]) = -\eta([X, Y]) = -\eta(N(X, Y)).$$



Meaning

$N = 0$ exactly when $T^{1,0}M$ is involutive under the Lie bracket.

There exist other conventions for the Nijenhuis tensor. They only differ by constants.

Identities from $d^2 = 0$ and equivalent conditions

Separating d^2 by bidegree gives, we get

$$\mu^2 = 0, \quad \mu\partial + \partial\mu = 0, \quad \partial^2 + \mu\bar{\partial} + \bar{\partial}\mu = 0 \quad \text{and} \quad \mu\bar{\mu} + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\mu}\mu = 0,$$

together with their conjugates.

If $\mu = \bar{\mu} = 0$, the above relations reduce to

$$d = \partial + \bar{\partial}, \quad \partial^2 = \bar{\partial}^2 = 0, \quad \partial\bar{\partial} + \bar{\partial}\partial = 0.$$

0

On an almost complex manifold, the following are equivalent:

- 1 $T^{1,0}M$ is involutive.
- 2 $\mu = \bar{\mu} = 0$.
- 3 $\bar{\partial}^2 = 0$ on all forms.

Newlander–Nirenberg and consequences

Theorem (Newlander-Nirenberg)

A smooth almost complex structure admits compatible local holomorphic coordinates if and only if its Nijenhuis tensor vanishes.

- Necessity follows in holomorphic coordinates: $T^{1,0}$ is spanned by commuting $\partial/\partial z_j$.
- Sufficiency is a **deep** analytic result.

Frobenius gives **smooth** coordinates. The hard part is to prove they are **analytic**.

- In real dimension two, the obstruction vanishes for dimension reasons. Every smooth almost complex surface is a Riemann surface.
- If $Z \subset X$ has J -invariant tangent spaces and X is complex, then the induced almost complex structure on Z is integrable.