

Complex Geometry Exercises

October 27, 2026

Exercise 1. *Prove the following theorems for holomorphic functions in n variables $f : U \rightarrow \mathbb{C}$ with $U \subseteq \mathbb{C}^n$:*

- (i) *Open mapping theorem*
- (ii) *Maximum principle*
- (iii) *(Generalised) Liouville theorem*

Exercise 2. *Prove that there is a one-to-one correspondence between holomorphic germs of $\mathcal{O}_{\mathbb{C}^n,0}$ and convergent power series in $\mathbb{C}[[z_1, \dots, z_n]]$.*

Exercise 3. *Finish the proof of the Weierstrass Division Theorem.*

Exercise 4. *Verify the following statements about analytic and holomorphic germs:*

- (i) *For any subset $A \subseteq \mathcal{O}_{X,x}$, $Z(A)$ is a well-defined analytic germ with $Z(A) = Z((A)_{\mathcal{O}_{X,x}})$.*
- (ii) *For every analytic germ Z , $I(Z) = \{f \in \mathcal{O}_{X,x} \mid Z \subset Z(f)\}$ is an ideal.*
- (iii) *If $X_1 \subset X_2$ are analytic germ, then $I(X_2) \subset I(X_1)$.*
- (iv) *If $I_1 \subset I_2$ are ideals in $\mathcal{O}_{X,x}$, then $Z(I_2) \subset Z(I_1)$.*
- (v) *$Z = Z(I(Z))$ and $I \subset I(Z(I))$.*
- (vi) *$Z(I \cdot J) = Z(I) \cup Z(J)$ and $Z(I + J) = Z(I) \cap Z(J)$.*

Exercise 5. *Prove the weak Nullstellensatz: If $f \in \mathcal{O}_{X,x}$ is irreducible and $g \in I(Z(f))$, then $f \mid g$.*

Exercise 6. *Let (M, J) be an almost complex manifold.*

(i) Show that $\dim_{\mathbb{R}} M$ is even and M is naturally oriented.

(ii) Show that there is at most one structure of a complex manifold inducing J .

Exercise 7. Show that the only compact connected complex submanifold in $\mathbb{C}^n$ is a point.

Exercise 8. Show that any Hopf surface

$$H_{\lambda}^2 := \mathbb{C}^2 \setminus \{(0, 0)\} / (z_1, z_2) \sim \lambda(z_1, z_2), \quad \lambda \in (0, 1)$$

contains many elliptic curves.

Exercise 9. Let $f : \mathbb{CP}^n \rightarrow \mathbb{C}^m / \Lambda$ be a holomorphic map. Then f is constant.

(**Hint:** think about how f interacts with the covering maps of the tours).

Exercise 10. Let (M, J) be an almost complex manifold and consider the associated ∂ and $\bar{\mu}$ operators. Prove that they satisfy the following properties:

(i) The Leibniz rule.

(ii) ∂ is $\mathbb{C}$ -linear and μ is function linear.

(iii) The following identities hold:

$$\begin{aligned} \mu\partial + \partial\mu &= 0, & \partial^2 + \bar{\partial}\mu + \mu\bar{\partial} &= 0, \\ \mu^2 &= 0, & \mu\bar{\mu} + \bar{\partial}\partial + \partial\bar{\partial} + \bar{\mu}\mu &= 0. \end{aligned}$$

Exercise 11 (Serge and Plücker embeddings).

(i) Show that there is a holomorphic embedding $\mathbb{CP}^n \times \mathbb{CP}^m \rightarrow \mathbb{CP}^{(n+1)(m+1)-1}$

(ii) Write down the corresponding homogeneous equations in the case $n = m = 1$.

(iii) Show that the map

$$\begin{aligned} Gr(k, n) &\rightarrow \mathbb{P}(\Lambda^k \mathbb{C}^n) \\ W = \text{span}\{w_1, \dots, w_k\} &\mapsto [w_1 \wedge \dots \wedge w_k] \end{aligned}$$

is a holomorphic embedding.

(iv) Write down the corresponding homogeneous equations for the Grassmannian $Gr(2, 4)$.

Exercise 12. Let $B = B_r(0) \subseteq \mathbb{C}^n$ be a polydisc. Then, the cohomology groups $H_{BC}^{p,q}(B)$ and $H_A^{p,q}(B)$ vanish for $p, q > 0$.

Exercise 13. Let X be a topological space, and A be an abelian group. Consider the presheaves

- $\mathcal{F}_1(U) = A$ for all open sets $U \subseteq X$,
- $\mathcal{F}_2(U) = A$ for all open sets $\emptyset \neq U \subseteq X$ and $\mathcal{F}_2(\emptyset) = 0$,

with the obvious restriction maps in all cases.

- (i) Show that neither $\mathcal{F}_1$ or $\mathcal{F}_2$ are sheaves.
- (ii) Show that they have the same étale space, given

$$\acute{E}t(\mathcal{F}^{const}) = X \times A$$

where A is endowed with the discrete topology, and so $\mathcal{F}_i^+ = \underline{A}$.

- (iii) Compute $\underline{A}(U)$ for a given open set U .

Exercise 14. Show that a sequence of sheaves

$$\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$$

is exact if and only if the corresponding sequence of stalks

$$\mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_x \xrightarrow{\beta_x} \mathcal{H}_x$$

is exact for all $x \in X$.

Exercise 15. Show that for a topological space $X, U \subset X$, a closed subset with inclusion map $j : U \hookrightarrow X$ and a sheaf $\mathcal{F}$ of abelian groups on U ,

$$H^q(U, \mathcal{F}) \cong H^q(X, j_*\mathcal{F})$$

for all $q > 0$.

Exercise 16. Let X be a complex manifold. We say a sheaf is a coherent sheaf if every point $x \in X$ has an open neighbourhood U in X and an exact sequence

$$\mathcal{O}_X^p|_U \rightarrow \mathcal{O}_X^q|_U \rightarrow \mathcal{F}|_U \rightarrow 0$$

for some natural numbers p and q , with $\mathcal{O}_X^k = \bigoplus_{i=1}^k \mathcal{O}_X$.

- (i) Show that the sheaf of sections of a holomorphic vector bundle is a coherent sheaf.
- (ii) Show that a coherent sheaf where we can take $p = 0$ for all points has an associated holomorphic bundle associated to it.

Let $\iota : Z \hookrightarrow X$ be a closed submanifold in X .

- (iii) Show that the ideal sheaf of Z , denoted by $\mathcal{I}_Z$, is coherent, where

$$\mathcal{I}_Z(U) = \left\{ f \in \mathcal{O}_X(U) \mid f|_Z = 0 \right\},$$

for $U \subseteq X$ open.

- (iv) Show that the direct image $\iota^* \mathcal{O}_Z$ is a coherent sheaf.
- (v) Prove that there is a short exact sequence of sheaves:

$$0 \rightarrow \mathcal{I}_Z \rightarrow \mathcal{O}_X \rightarrow \iota^* \mathcal{O}_Z \rightarrow 0.$$

Exercise 17. Show that

$$H^q(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \cong \begin{cases} 0 & \text{for } q \neq 0 \\ \mathbb{C} & \text{for } q = 0 \end{cases}.$$

Exercise 18. Prove that, up to vector bundle isomorphism, we have the following correspondences:

- real vector bundles of rank $r \xleftarrow{1:1} \check{H}^1(X, \text{GL}(r, \mathcal{C}^\infty(X, \mathbb{R})))$,
- complex vector bundles of rank $r \xleftarrow{1:1} \check{H}^1(X, \text{GL}(r, \mathcal{C}^\infty(X, \mathbb{C})))$
- holomorphic vector bundles of rank $r \xleftarrow{1:1} \check{H}^1(X, \text{GL}(r, \mathcal{O}_X))$,

where $\text{GL}(r, \mathcal{F})$ is the sheaf of invertible rank k matrices with coefficients in the sheaf $\mathcal{F}$.

Exercise 19. Show that any short exact sequence of holomorphic vector bundles

$$0 \longrightarrow L \longrightarrow E \longrightarrow F \longrightarrow 0,$$

where L is a line bundle, induces the short exact sequence

$$0 \longrightarrow L \otimes \Lambda^{k-1} E \longrightarrow \Lambda^k E \longrightarrow \Lambda^k F \longrightarrow 0.$$

Exercise 20. Let L_1 and L_2 be holomorphic line bundles such that $L_1|_{X \setminus Y} = L_2|_{X \setminus Y}$ for $Y \subset X$ a complex submanifold of codimension ≥ 2 . Show that $L_1 \cong L_2$.

Exercise 21. Prove that for a 1-dimensional connected complex manifold X ,

$$K(X) \cong \{f : X \rightarrow \mathbb{P}^1 \mid f \text{ is holomorphic, } f \not\equiv \infty\}.$$

Exercise 22. Let $X = \mathbb{C}^n/\Lambda$ be a complex torus. Show that:

(a) $\tau_X = \mathcal{O}_X^n$, $K_X = \mathcal{O}_X$

(b) $\text{kod}(X) = 0$ and $\text{kod}(Y) \geq 0$ for every (positive-dimensional) submanifold $Y \subset X$

Exercise 23 (Veronese embedding). Show that the map defined by global sections $H^0(\mathbb{P}^1, \mathcal{O}(2))$ is an embedding

$$\phi : \mathbb{P}^1 \hookrightarrow \mathbb{P}^2$$

realising $\mathbb{P}^1$ as the zero set of a homogeneous polynomial of degree 2.

Exercise 24. Assume X is a connected complex manifold.

(i) Show that a holomorphic line bundle L over X is isomorphic to the trivial bundle if and only if both L and L^* admit a non-trivial global section.

(ii) Deduce $H^0(\mathbb{P}^n, \mathcal{O}(k)) = 0$ for $k < 0$.

(iii) Prove (or convince yourself) that there exist line bundles such that

$$H^0(X, L) = 0 = H^0(X, L^*) .$$

Exercise 25. Show that the following hypersurfaces are smooth and compute their Kodaira dimension:

(a) $Z(x_0^2 + z_1^2 + z_2^2) \subset \mathbb{CP}^2$

(b) $Z(x_0^3 + z_1^3 + z_2^3) \subset \mathbb{CP}^2$

(c) $Z(x_0^5 + \cdots + z_5^5) \subset \mathbb{CP}^4$

Exercise 26. Show that the image of $\text{Div}(X) \rightarrow \text{Pic}(X)$ is given by the classes of line bundles which admit a non-zero meromorphic section.

Exercise 27 (Bezout's theorem). Let $C, D \subset \mathbb{P}^2$ smooth (distinct) curves defined by homogeneous polynomials f and g of degrees d and e respectively.

(i) Show that the line bundle $\mathcal{O}(1)$ restricted to C is of degree d .

(ii) Show that

$$d \cdot e = \sum_{p \in C \cap D} \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^2, g} / (f, g).$$

Exercise 28. Let X be a smooth complex manifold with $K_X \cong \mathcal{O}_X$. Show that X cannot be obtained by a blow-up of another surface.

Exercise 29. Let X be a smooth Calabi-Yau manifold (with $K_X \cong \mathcal{O}_X$). Show that X cannot be obtained by a blow-up of another surface.

Exercise 30. For all $n > 0, k \in \mathbb{Z}$, compute

$$H^n(\mathbb{CP}^n, \mathcal{O}(k)) .$$

Exercise 31. Let $(V^{2n}, \langle \cdot, \cdot \rangle)$ be a Euclidean vector space.

(i) Show that the space of compatible complex structure is parametrised by two disjoint copies of $\mathrm{SO}(2n)/\mathrm{U}(n)$.

(ii) Show that for $n = 2$, this corresponds to two copies of $\mathbb{CP}^1$.

(iii) Show that for $n = 3$, this corresponds to two copies of $\mathbb{CP}^3$.

(**Hint:** Recall the exceptional isomorphisms $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$ and $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$.)

For the remainder of the sheet, let $(V^{2n}, \langle \cdot, \cdot \rangle, I)$ denote a Euclidean vector space of real dimension $2n$ with compatible complex structure I .

Exercise 32. Let $p, q, p', q' \in \mathbb{N}$ with $p + q = k = p' + q'$ and $k \leq n$.

Consider the Hodge-Riemann pairing:

$$Q : \bigwedge^{p,q} V_{\mathbb{C}}^* \times \bigwedge^{p',q'} V_{\mathbb{C}}^* \rightarrow \mathbb{C}$$

$$(\alpha, \beta) \mapsto (-1)^{\binom{k}{2}} \alpha \wedge \beta \wedge \omega^{n-k} .$$

Show

(i) Q vanishes unless $(p, q) = (q', p')$.

(ii) For $0 \neq \alpha \in P^{p,q} \subseteq \Lambda^{p,q} V_{\mathbb{C}}^*$, we have

$$i^{p-q} Q(\alpha, \bar{\alpha}) = [n - (p + q)]! \langle \alpha, \alpha \rangle > 0 .$$

Exercise 33. Let $x_1, y_1 = I(x_1), \dots, x_n, y_n = I(x_n)$ be an orthonormal basis of V . Show that, for any $\alpha \in \Lambda^k V$

$$\Lambda\alpha(X_1, \dots, X_{k-2}) = \sum_{i=1}^n \alpha(x_i, y_i, X_1, \dots, X_{k-2}) .$$

Exercise 34. Let $(E, \bar{\partial}_E, h)$ be a holomorphic hermitian vector bundle on X . Show that

- (i) the space of connections on E is an affine space modelled on $\mathcal{A}_X^1(\text{End}(E))$;
- (ii) the space of metric connections on E is an affine space modelled on $\mathcal{A}_X^1(\text{End}(E, h))$;
- (iii) the space of compatible connections on E is an affine space modelled on $\mathcal{A}_X^{1,0}(\text{End}(E))$.

Use the above to give an alternate proof of the uniqueness of the Chern connection.

Exercise 35. Prove that

- (i) $\bar{\partial}^* = - * \partial *$ and $\partial^* = - * \bar{\partial} *$,
- (ii) $\Delta_{\bar{\partial}} = \overline{\Delta_{\partial}}$.
- (iii) $\mathcal{H}_{\bar{\partial}}^{p,q} = \ker \bar{\partial} \cap \ker \bar{\partial}^*$.

Exercise 36. The goal of this exercise is to establish a Hodge isomorphism for vector bundles.

Let $(E, \bar{\partial}_E, h)$ be a holomorphic hermitian vector bundle.

- (i) Show that the Hodge star extends naturally to E -valued forms.
- (ii) Show that $\bar{\partial}_E^* = - * \bar{\partial}_E *$ is an L^2 -adjoint to $\bar{\partial}_E$ with respect to the L^2 -inner product induced by h .
- (iii) Assuming we have an analogous Hodge decomposition, reproduce the argument in the lectures to conclude there is an isomorphism

$$H^q(X, \Omega^p \otimes E) \cong \mathcal{H}^{p,q}(X, E) ,$$

where $\mathcal{H}^{p,q}(X, E) := \ker \Delta_{\bar{\partial}_E}$.

Exercise 37. Let (X, g, J) be a Kähler manifold and $\iota : Y \hookrightarrow X$ a closed complex submanifold. Use the Wirtinger inequality to prove that Y is volume-minimising within its homology class.

Exercise 38. Consider the open unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. Prove that the $(1,1)$ -form

$$\omega = -\frac{i}{2} \partial \bar{\partial} \log(1 - |z|^2)$$

defines a Kähler metric on $\mathbb{D}$.

Exercise 39. The goal of the exercise is to prove that the Iwasawa manifold $\mathbb{I} = \mathbb{U}/\mathbb{U}_{\mathbb{Z}}$ is not Kähler.

- (i) Prove that holomorphic forms on a Kähler are harmonic.
- (ii) Compute a basis of left-invariant holomorphic 1-forms for $\mathbb{I}$ and conclude it is a weak Calabi-Yau manifold.
- (iii) Deduce it does not admit a compatible Kähler metric.

Exercise 40. Consider the hermitian metric

$$h_z(s, s') = \frac{\langle s, s' \rangle}{\sum_i |z_i|^2}$$

on $\mathcal{O}(1)$, induced by its global sections. Denote by ∇ its Chern connection. Prove

- (i) $\frac{i}{2\pi} F_{\nabla} = \omega_{FS}$, where ω_{FS} is the Kähler form of the Fubini-Study metric.
- (ii) $\int_{\mathbb{CP}^n} \omega_{FS}^n = 1$

Exercise 41. Show that, in dimension ≥ 2 , there exists at most one Kähler metric in a given conformal class (up to scale).

Exercise 42. Let $\text{Pic}^0(X) := \ker(\text{Pic}(X) \rightarrow H^2(X, \mathbb{Z}))$.

- (i) Show that if X is Kähler is a complex torus of dimension $b^1(X)$.
- (ii) Give a counterexample when X is not Kähler.

Exercise 43.

- (i) Compute the signature of a closed complex surface S with $\kappa(S) = -\infty$.
- (ii) Prove that $\tilde{S}_{2k} = \#_{2k} \mathbb{CP}^2$ cannot be Kähler for $k \geq 1$.

Exercise 44. Consider the complex torus $\mathbb{T}_{\Gamma} = \mathbb{C}^2/\Gamma$ for a lattice Γ .

- (i) Find Γ such that the only line bundle in $\mathbb{T}_\Gamma$ that admits a holomorphic structure is the trivial one.
- (ii) Compute its Picard group, $\text{Pic}(\mathbb{T}_\Gamma)$.
- (iii) Prove (or convince yourself) that this is in fact true for a generic Γ .
- (iv) Conclude that most tori in dimensions ≥ 2 are not projective.

Exercise 45. Let X be a complex manifold such that $h^{2,0}(X) = 0$. Show that X is projective. Conclude that Calabi-Yau manifolds (in the strict sense) of dimension ≥ 3 are all projective.

Exercise 46. Let X be a Kähler manifold, with $\dim(X) \geq 4$, and consider $\iota : Y \rightarrow X$ a hypersurface with $\mathcal{O}(Y)$ positive. Assuming $H^2(X, \mathbb{Z})$ and $H^2(Y, \mathbb{Z})$ are torsion-free, prove that $\iota^* : \text{Pic}(X) \rightarrow \text{Pic}(Y)$ is an isomorphism.

Exercise 47. Let Σ be a Riemann surface. Prove that

$$c_1(K_\Sigma) = 2g(\Sigma) - 2.$$

(**Recall:** For a conformal metric g , the Gaussian curvature is $K = \Delta \log(g)$.)

Exercise 48. Let Σ_d be a smooth hypersurface in $\mathbb{CP}^2$. Show that

$$g(\Sigma_d) = \frac{(d-1)(d-2)}{2}.$$

Exercise 49. Let X, Y be closed complex manifolds. Prove that they are projective if and only if their product is. Give two alternative proofs, using

- (i) the Serre embedding, and
- (ii) the Kodaira embedding theorem.

Exercise 50. Let $L \rightarrow \Sigma$ be a line bundle on a Riemann surface Σ . Prove the following assertions.

- (i) If $\deg(L) < 0$, then $H^0(\Sigma, L) = 0$.
- (ii) If $\deg(L) > \deg(K_\Sigma)$, $H^1(\Sigma, L) = 0$.
- (iii) If $\deg(L) > \deg(K_\Sigma) + 2$, its pluricanonical map ϕ_L is an embedding.

In particular, all Riemann surfaces are projective.

Exercise 51. *Prove that the blow-up of a projective manifold is again projective.*

Exercise 52. *Show that any vector bundle E on a projective manifold X can be written as a quotient $(L^k)^{\oplus l} \rightarrow E$ with L an ample line bundle, $k \ll 0$ and $l \gg 0$.*

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Exercise 54. *Prove that a complex torus $\mathbb{T}_\Gamma = \mathbb{C}^n/\Gamma$ is projective if and only if there exists an alternating bilinear form*

$$\omega : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{R}$$

such that

- $\omega(iu, iv) = \omega(u, v)$,
- $\omega(\cdot, i \cdot)$ is positive definite, and
- $\omega(u, v) \in \mathbb{Z}$ for all $u, v \in \Gamma$

Exercise 55. *Let X be a compact Kähler manifold and*

$$\text{Alb}(X) := H^0(X, \Omega_X)^* / H_1(X, \mathbb{Z})$$

where

$$H_1(X, \mathbb{Z}) \rightarrow H^0(X, \Omega_X)^*, \quad \gamma \mapsto \left(\alpha \mapsto \int_\gamma \alpha \right).$$

Prove the following statements.

- (i) $\text{Alb}(X)$ is a complex torus.
- (ii) Fixing a base point $z_0 \in X$, yields a holomorphic map $X \rightarrow \text{Alb}(X)$.
- (iii) For $\mathbb{T}_\Gamma = \mathbb{C}^n/\Gamma$, the map $\mathbb{T}_\Gamma \rightarrow \text{Alb}(\mathbb{T}_\Gamma)$ is a biholomorphism.

Exercise 56. *Let (X, J) be a complex manifold. Show that $H^0(X, \tau_X)$ is naturally a complex Lie algebra, for which the Lie bracket is complex bilinear.*

(Hint: Recall the Nijenhuis tensor $N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y].$)

Exercise 57. *Prove the following assertions.*

- (i) The group $\text{Aut}(\mathbb{CP}^n)$ is path-connected.
- (ii) Any $f \in \text{Aut}(\mathbb{CP}^n)$, with $f \neq 1$, has $n + 1$ fixed points (counted with multiplicity).
- (iii) The group $\text{Aut}(\mathbb{CP}^1)$ is sharply 3-transitive.
- (iv) The group $\text{Aut}(\mathbb{CP}^n)$ is 2-transitive but not 3-transitive for $n \geq 2$.

Exercise 58.

- (i) Consider (E_1, h_1) and (E_2, h_2) holomorphic hermitian vector bundles over a Kähler manifold. Prove that

$$\mathcal{H}^{p,q}(E_1 \oplus E_2) \cong \mathcal{H}^{p,q}(E_1) \oplus \mathcal{H}^{p,q}(E_2) .$$

- (ii) Using the Euler sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{CP}^n} \rightarrow \bigoplus_{i=0}^n \mathcal{O}(1) \rightarrow \tau_{\mathbb{CP}^n} \rightarrow 0 ,$$

show that

$$\dim H^i(\mathbb{CP}^n, \tau_{\mathbb{CP}^n}) = \begin{cases} (n+1)^2 - 1 & \text{if } i = 0 , \\ 0 & \text{otherwise} \end{cases} .$$

Exercise 59. Let $f : X \rightarrow X$ be a continuous map. We say z is fixed by f if $f(z) = z$. For an isolated fix point, we define the index of z as the degree of the homology map induced in $U \setminus \{z\}$ for U a small enough neighbourhood of z .

Assume that f is holomorphic.

- (i) Show that the degree is always positive.
- (ii) Using local coordinates, show that the degree is equal to the multiplicity of the zero of the (locally defined) function $g(z) = f(z) - z$.

Assume that X is closed.

- (iii) Prove that if $f \neq \text{Id}$, the number of fixed points (with multiplicity) coincides with the Lefschetz number of f :

$$\Lambda_f = \sum_{k \geq 0} \text{Tr} \left(f^* : H^k(X, \mathbb{R}) \rightarrow H^k(X, \mathbb{R}) \right) .$$

- (iv) Compute the number of fixed points if f is homotopic to the identity.

(v) Study the fixed points of $f \in \text{Aut}(\mathbb{T}^n)$.

Exercise 60. Prove that the standard identity

$$d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])$$

extends to vector-valued 1-forms ϕ :

$$d\phi(X, Y) = [X, \phi(Y)] - [Y, \phi(X)] - \phi([X, Y]) .$$

Exercise 61. Consider $H_\lambda = \mathbb{C}^n \setminus \{0\} / \langle \lambda \rangle$ the Hopf surface, with $\lambda \in \mathbb{C}^*$, $|\lambda| < 1$, for $n \geq 2$.

- (i) Show that $f \in \text{Aut}(H_\lambda)$ extends to a biholomorphism $\tilde{f} : \mathbb{C}^n \rightarrow \mathbb{C}^n$ with $\tilde{f}(0) = 0$.
- (ii) Show that $\tilde{f}(\lambda z) = \lambda \tilde{f}(z)$ (as opposed to $\lambda \tilde{f}(\lambda z) = \tilde{f}(z)$).
- (iii) Prove that $\partial \tilde{f}(\lambda^n z) = \partial \tilde{f}(z)$ for all $n \in \mathbb{Z}$, so $\tilde{f} \in \text{GL}(n, \mathbb{C})$.
- (iv) Compute $\text{Aut}(H_\lambda)$.

Exercise 62. Let X be a Kähler manifold with $c_1(K_X) < 0$. Prove that $H^2(X, \tau_X) = 0$.

Exercise 63. Consider a flat torus $\mathbb{T}^n = \mathbb{C}^n / \Gamma$.

- (i) Find an explicit basis of $\mathcal{H}^{p,q}(\mathbb{T}^n)$. Show that $h^{p,q} = \binom{n}{p} \binom{n}{q}$.
- (ii) Compute the virtual dimension of complex structures $\mathcal{M}_{\mathbb{T}^n}$.
- (iii) Show that the obstruction map $H^1(\mathbb{T}^n, \tau_{\mathbb{T}^n}) \xrightarrow{\Phi} H^2(\mathbb{T}^n, \tau_{\mathbb{T}^n})$ vanishes.
- (iv) Study how the group $\text{Aut}^0(\mathbb{T}^n)$ acts on $H^1(\mathbb{T}^n, \tau_{\mathbb{T}^n})$.
- (v) Compute the dimension of $\mathcal{M}_{\mathbb{T}^n}^0 := \Psi^{-1}(0) / \text{Aut}^0(\mathbb{T}^n)$.

Exercise 64 (Moduli space of complex tori).

- (i) Prove that a complex torus of dimension n is determined by a map $\mathbb{Z}^{2n} \rightarrow \mathbb{C}^n \cong \mathbb{R}^{2n}$, and thus, can be identified with an element of $\text{GL}(2n, \mathbb{R})$.

So there is a surjective map $\text{GL}(2n, \mathbb{R}) \rightarrow \mathcal{M}_{\mathbb{T}^n}$.

- (ii) Prove the actions of $\text{GL}(n, \mathbb{C})$ and $\text{GL}(2n, \mathbb{Z})$ on $\text{GL}(2n, \mathbb{R})$ induce the trivial action on $\mathcal{M}_{\mathbb{T}^n}$.
- (iii) Conclude that $\text{GL}(n, \mathbb{C}) \setminus \text{GL}(2n, \mathbb{R})$ is a "covering" space of $\mathcal{M}_{\mathbb{T}^n}$.

(iv) Prove that $\mathcal{M}_{\mathbb{T}^n}$ is isomorphic to the bi-quotient

$$\mathrm{GL}(n, \mathbb{C}) \setminus \mathrm{GL}(2n, \mathbb{R}) / \mathrm{GL}(2n, \mathbb{Z}) .$$

(v) Prove that $\mathrm{GL}(2n, \mathbb{Z})$ does not act properly discontinuous on $\mathrm{GL}(n, \mathbb{C}) \setminus \mathrm{GL}(2n, \mathbb{R})$ for $n \geq 2$, so $\mathcal{M}_{\mathbb{T}^n}$ is not Hausdorff.

(vi) For $n = 1$, prove that

$$\mathrm{GL}(1, \mathbb{C}) \setminus \mathrm{GL}(2, \mathbb{R}) \cong \mathbb{H}^2 \sqcup \mathbb{H}^2 ,$$

Where $\mathbb{H}^2 = \{x + iy \in \mathbb{C} \mid y > 0\}$ is the hyperbolic plane, and the two copies are distinguished by orientation.

(vii) Give a geometric description of $\mathcal{M}_{\mathbb{T}^1}$.

(viii) Show that $\mathcal{M}_{\mathbb{T}^1}$ is not compact. Can you find a natural compactification?

(ix) Show that $\mathcal{M}_{\mathbb{T}^1}$ is not a manifold, but an orbifold. Can you compute the stabiliser of the non-smooth points?

Exercise 65 (Moduli space of principally polarised tori). From Exercise 6 Sheet 9, we know that a complex torus $\mathbb{C}^n / \Gamma$ is projective iff ω restricts to a symplectic form $\Gamma \times \Gamma \rightarrow \mathbb{Z}$. Fix a symplectic structure ω (equivalently a hermitian metric) on $\mathbb{C}^n$.

(i) Prove that a projective torus of dimension n is determined by a symplectic map $\mathbb{Z}^{2n} \rightarrow \mathbb{C}^n \cong \mathbb{R}^{2n}$, and thus, can be identified with an element of $\mathrm{Sp}(2n, \mathbb{R})$.

We define $\mathcal{M}_{Ab}^n$ the moduli of principally polarised n -dimensional tori to be the collection of tori given by the construction in (i), modulo automorphism.

(ii) Show the actions of $U(n)$ and $\mathrm{Sp}(2n, \mathbb{Z})$ on $\mathrm{Sp}(2n, \mathbb{R})$ induce the trivial action on $\mathcal{M}_{Ab}^n$.

(iii) Prove that $\mathcal{M}_{Ab}^n$ is isomorphic to the bi-quotient

$$U(n) \setminus \mathrm{Sp}(2n, \mathbb{R}) / \mathrm{Sp}(2n, \mathbb{Z}) .$$

(iv) Check that, for $n = 1$, we have $\mathcal{M}_{Ab}^1 \cong \mathcal{M}_{\mathbb{T}^1}$.

(v) Prove that $\mathrm{Sp}(2n, \mathbb{Z})$ acts properly discontinuously on $U(n) \setminus \mathrm{Sp}(2n, \mathbb{R})$. In particular, the moduli space $\mathcal{M}_{Ab}^n$ is Hausdorff for all $n \geq 2$.

Exercise 66. For $\omega_{FS} = \frac{i}{2\pi} \partial \bar{\partial} \log(1 + |z|^2)$ on $\mathbb{CP}^n$, compute its Ricci form and verify

$$\mathrm{Ric}(\omega_{FS}) = 2\pi(n+1)\omega_{FS}.$$

Compare its cohomology class with the Euler sequence.

Exercise 67. Let X be a compact Riemann surface. Use the identity $\omega_\varphi = (1 + \Delta_\omega \varphi)\omega$ (where $i\partial\bar{\partial}\varphi = (\Delta_\omega \varphi)\omega$) to solve the prescribed-volume equation, and explain why this linear equation is different from prescribing constant Gaussian curvature when the Euler characteristic is non-zero.

Exercise 68. Show directly that the linearization at $\varphi = 0$ of the map $\varphi \mapsto \omega_\varphi^n$ is $n i\partial\bar{\partial}\varphi \wedge \omega^{n-1}$. Verify that its integral vanishes, and relate this to the auxiliary constant in the openness argument.

Exercise 69. Prove that a holomorphic automorphism preserving a Kähler class on a compact Ricci-flat Kähler manifold preserves the unique Ricci-flat metric in that class. Apply this to translations on a complex torus.