Lecture Notes on Riemannian Geometry and Holonomly

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Contents

These lecture notes originate from a graduate course I taught in the spring of 2026 at the Yau Mathematical Sciences Centre (YMSC) in Tsinghua University, during my tenure as a Chern Instructor at the Beijing Institute of Mathematical Sciences and Applications (BIMSA).

The main goal of the course is to provide a gentle introduction to the vast subject of holonomy and its geometric interpretation, specially in connection with Riemanninan geometry. The main result covered in detail in the first half of the notes is the Berger holonomy classification, which characterises which Lie groups can be realised as "interesting" holonomy groups of a Riemannian metric on a complete manifold.

The notes are based on the classical books of Shoshichi Kobayashi and Katsumi Nomizu, [SN96a] and [SN96b], Arthur Besse [Bes87], and the monograph of Dominic Joyce [Joy00]. The reader is also encouraged to consult the abridged book of Joyce [Joy07]. Some contents have been drawn from published research articles, most notably the seminal papers of Bryant [Bry87], Bryant and Salamon [BS89], Guillemin [Gui65], Olmos [Olm05], and Stenzel [Ste93], amongst others.

There is (almost) no discussion on the special geometric objects supported on special holonomy manifolds, namely calibrated submanifolds and instantons, due to time constraints.

I am convinced that there are a (n embarrassingly large) number of typos and minor mistakes in these notes. I profusely apologise for that. If you find any, please let me know by dropping me an email at esolefarre@bimsa.cn.

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Overview

Our study of special holonomy is motivated by questions in geometric analysis concerning the existence of canonical metrics. For a given smooth manifold M, one may ask whether there exists a natural, or preferred, Riemannian metric on it. Geometric analysis addresses this problem by seeking a metric that satisfies a specific partial differential equation (or inequality) derived from its curvature.

Classic examples involve the Riemann curvature tensor Rm_g and its contractions, leading to second-order elliptic PDEs (modulo diffeomorphisms), such as:

$$\mathrm{Rm}_g = 0, \qquad \mathrm{Ric}_g = \lambda g, \qquad \sec_g = \lambda \, \mathrm{Id} \qquad \text{and} \qquad \mathrm{scal}_g = \lambda \; ,$$

for $\lambda \in \mathbb{R}$. By contrast, metrics with special holonomy are characterised by first-order differential conditions, which arise from underlying algebraic structures on M. However, as we will see, these first-order conditions typically imply that the metric also satisfies one of the second-order PDE above, without depending explicitly on the underlying aglberaic structure.

The study of metrics of special holonomy sits at the intersection of differential geometry, global analysis, and representation theory. The idea of holonomy is simple: given a connection on a vector bundle, parallel transport along loops defines a transformation of the fibre at the base point. The collection of all such transformations forms a group, known as the holonomy group. This holonomy group encodes essential geometric information about the manifold and its connection.

The concept of holonomy and its study dates back to Élie Cartan's work in the early 20th century. In 1952, Marcel Berger, a student of Cartan, classified the possible holonomy groups of Riemannian manifolds under the assumptions of irreducibility and non-symmetry.

Hol(g)	**dim(M)**	**Type of manifold**	**Curvature**
$\mathrm{SO}(n)$	n	Orientable	–
$\mathrm{U}(n)$	$2n$	Kähler	–
$\mathrm{SU}(n)$	$2n$	Calabi–Yau	Ricci-flat
$\mathrm{Sp}(n)$	$4n$	Hyperkähler	Ricci-flat
$\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$	$4n$	Quaternion-Kähler	Einstein $\lambda \neq 0$
G_2	7	Holonomy G_2	Ricci-flat
Spin(7)	8	Holonomy Spin(7)	Ricci-flat

Table 1: Berger's list of possible holonomy groups

Berger's list included all the special holonomy groups we study today, though it took several decades to fully understand their geometric significance and construct examples, as well as rule out a leftover case from Berger's original work.

The study of special holonomy metrics had a resurgence in the late 1970s with Yau's proof of the Calabi conjecture, followed by Bryant's discovery of (local) metrics with holonomy G_2 in 1987, the construction of the first complete (non-compact) examples by Bryant and Salamon in 1989 and the subsequent construction of compact examples by Joyce in the mid-1990s.

With the work of Joyce, we now have examples of compact and non-compact complete manifolds whose holonomy groups cover the list found by Berger, which will be discussed towards the end of these notes.

The course is organised as follows. The first four sections focus on the existence of general geometric structures on manifolds from the point of view of pseudogroups. In doing so, we introduce (or revise for the advanced reader) the notions of principal bundles and connections, and all the related notions.

Sections five to nine focus on restricting the previous discussion to the case where the different geometric structures are compatible with a Riemannian metric. The main goal of the last three sections is to provide two comprehensive proofs of Berger's theorems.

Sections ten to twelve are dedicated to constructing special holonomy metrics using different methods, on compact and non-compact manifolds.

1 Pseudogroups, Lie groups and representations

We begin by reviewing fundamental definitions from smooth manifolds and some basic results in Lie theory. Throughout, M will denote a topological manifold: a second-countable, locally Euclidean topological space. These conditions imply that M is paracompact and Hausdorff. In particular, every open cover of M admits a subordinate partition of unity (cf. [Mun00, Thm. 41.7]). We define

Definition 1.1. A $\mathcal{C}^k$*-manifold* is a topological manifold equipped with an atlas of charts $(U_i, \phi_i)_{i \in I}$, where transition functions $\phi_{ij} = \phi_i \circ \phi_j^{-1}$ are $\mathcal{C}^k$-diffeomorphisms between open sets in $\mathbb{R}^n$.

To avoid issues and pathologies, we will always assume our atlases are maximal, i.e. they are not a proper subset of any other atlas. Every atlas $\{(U_i, \phi_i) : i \in I\}$ is contained in a unique maximal atlas: the set of all charts (U, ϕ) compatible with (U_i, ϕ_i) for all $i \in I$, so there is no prejudice in always taking the maximal atlas.

Note that a $\mathcal{C}^0$-manifold is simply a topological manifold, without any further structure. Moreover, a theorem of Whitney tells us that a $\mathcal{C}^k$-manifold for $k \geq 1$ admits a compatible $\mathcal{C}^\infty$-structure.

Between the topological category of continuous ($\mathcal{C}^0$) structures and the smooth category of differentiable ($\mathcal{C}^1$) manifolds lies an intermediate concept: piecewise-linear (PL) structures. Intuitively, a PL manifold is any topological manifold that may be triangulated, that is, decomposed into simplices (generalised triangles) glued together along their faces in a linear way. Therefore, the category of PL manifolds provides a combinatorial framework that is more rigid than pure topology but more flexible than smooth geometry, allowing techniques from both disciplines to be applied.

Understanding when a manifold admits a smooth structure, and if so, how many, was an active research area in the second half of the 20th century that is nowadays well understood (e.g. Kervaire–Milnor groups, Kirby–Siebenmann invariants, geometrisation conjecture) except in dimension 4, where surprising links to other areas of mathematics appear (e.g. Donaldson diagonalisation theorem, Seiberg–Witten theory).

The definition above can be generalised using the notion of pseudogroups:

Definition 1.2. A *pseudogroup of transformations* on $\mathbb{R}^n$ is a set Γ of local diffeomorphisms satisfying:

(i) Each $f \in \Gamma$ is a diffeomorphism between open subsets of $\mathbb{R}^n$.

(ii) For every open subset $U \subset \mathbb{R}^n$, $\mathrm{id}_U \in \Gamma$.

(iii) If $f \in \Gamma$, then $f^{-1} \in \Gamma$.

(iv) Let $U = \bigcup_i U_i$, with U_i open in $\mathbb{R}^n$. A diffeomorphism f with domain U belongs to Γ if and only $f|_{U_i} \in \Gamma$.

(v) If $f \colon U \to V$ and $f' \colon U' \to V'$ are in Γ with $V \cap U' \neq \emptyset$, $f' \circ f \in \Gamma$.

(vi) For every $p, q \in \mathbb{R}^n$ there exists $f \in \Gamma$ satisfying $f(p) = q$.

A Γ-atlas of M is an atlas of charts $(U_i, \phi_i)_{i\in I}$ of M such that $\phi_i \circ \phi_j^{-1} \in \Gamma$ for all $i, j \in I$.

We say that a manifold M equipped with a Γ-atlas is a Γ-manifold. We immidately see that the perspective of pseudogroups covers the case of $\mathcal{C}^k$-manifolds. Indeed a $\mathcal{C}^k$-manifold is a Γ-manifold, for $\Gamma = \mathrm{Diff}^k(\mathbb{R}^n)$, the collection of homeomorphisms between open sets of $\mathbb{R}^n$ of class $\mathcal{C}^k$, with inverse in $\mathcal{C}^k$. However, there are other examples of interest. For example, let $\mathrm{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \mathrm{GL}(n,\mathbb{R})$ be the group of affine transformations. Then, $\mathrm{Aff}(\mathbb{R}^n)$ is a pseudogroup of transformations of $\mathbb{R}^n$, and the corresponding Γ-manifolds are known as affine manifolds.

There is a particular class of Γ-manifolds that will play a motivating role:

Definition 1.3. Let G be a Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$. The *$G$-pseudogroup*, Γ_G, is the pseudogroup of local smooth (i.e. $\mathcal{C}^\infty$) diffeomorphisms of $\mathbb{R}^n$ whose Jacobian $D(\phi_i \circ \phi_j^{-1})$ lies in $G \subset \mathrm{GL}(n,\mathbb{R})$ at every point.

We give a few examples of interest of Γ_G-manifolds:

Example 1.4.

- *Let* $\mathrm{GL}_+(n,\mathbb{R})$ *be the subgroup of positivie determinant matrices.* $\Gamma_{\mathrm{GL}_+(n,\mathbb{R})}$*-manifolds are oriented smooth manifolds.*
- *Let* $\mathrm{SL}(n,\mathbb{R})$ *be the subgroup of matrices of determinant one.* $\Gamma_{\mathrm{SL}(n,\mathbb{R})}$*-manifolds are smooth manifolds equipped with a volume form.*
- *Let* $\mathrm{GL}(n,\mathbb{C}) \subseteq \mathrm{GL}(2n,\mathbb{R})$ *be the subgroup of complex-linear matrices.* $\Gamma_{\mathrm{GL}(n,\mathbb{C})}$*-manifolds are complex manifolds.*
- *Let* $\mathrm{GL}(\mathbb{Z}, n)$ *be the subgroup of integer-valued invertible matrices.* $\Gamma_{\mathrm{GL}(\mathbb{Z},n)}$*-manifolds are integral affine manifolds.*

Note that although the last example is a subclass of the class of affine manifolds, one cannot frame general affine manifolds as a Γ_G-structure.

It is clear that if we have an inclusion of Lie subgroups $G' \subseteq G$, it induces an inclusion of pseudogroups of transformations $\Gamma_{G'} \subseteq \Gamma_G$. A natural question is then whether a manifold M admits a Γ_G-structure, for a given Lie subgroup G. For instance, if $G = \mathrm{GL}_+(n,\mathbb{R})$, there is only a topological obstruction for a smooth manifold to admit a $\Gamma_{\mathrm{GL}_+(n,\mathbb{R})}$-structure. Similarly, there is no obstruction to go from a $\Gamma_{\mathrm{GL}_+(n,\mathbb{R})}$-manifold to a $\Gamma_{\mathrm{SL}(n,\mathbb{R})}$-manifold.

The case of $\Gamma_{\mathrm{GL}(n,\mathbb{C})}$-manifolds is much more involved, and to date, there are no good necessary and sufficient conditions for when a manifold admits a complex structure. In the compact setting, some existence and classification results exist for complex dimensions 1 and 2. Already in dimension 3, we find one of the most (in)famous open problems in differential geometry:

Question 1.5. *Does the round 6-sphere* S^6 *admit the structure of a complex structure?*

In the non-compact case, we have Liouville-type obstructions, so we know that the complex plane $\mathbb{C}^n$ is not biholomorphic to certain bounded domains (e.g. the unit ball or polydisc). However, there is no high-dimensional analogue of the Uniformisation Theorem. In general, complex domains carry intrinsic complex-analytic invariants that obstruct biholomorphism, and for $n > 1$, many bounded domains in $\mathbb{C}^n$ are not biholomorphically equivalent to each other. In particular, we have

Theorem 1.6 (Poincaré 1907)**.** *In $\mathbb{C}^n$ with $n > 1$, the n-ball $B^n := \{z \in \mathbb{C}^n \big| \, \|z\| < 1\}$ and the polydisk $D^n := \{z \in \mathbb{C}^n \big| \, |z_i| < 1\}$ are not biholomorphically equivalent.*

Poincaré's proof idea was to compute the automorphism groups of the two spaces that fix the origin. It is a standard complex analysis computation to show that

$$\mathrm{Aut}^0(B^n) \cong \mathrm{PSU}(n,1) \qquad\qquad \mathrm{Aut}^0(D^n) \cong (PSU(1,1))^n \rtimes S_n \ ,$$

from which Poincaré's observation follows.

Our main goal for this part of the course wi will be to try to understand when a manifold admits a Γ_G-structure for a given G.

We split the problem into two parts. First, we have an algebraic obstruction, detailed in Section 2. Then, in Section 4, we describe an infinite tower of tensor-like differential obstructions. First, we revise some basic results on Lie groups.

1.1 Lie groups

We recall some basic notions from Lie theory.

Definition 1.7. A *Lie group G* is a smooth manifold equipped with a group structure such that the group operations, multiplication and inversion, are smooth. A map $f : G_1 \to G_2$ between Lie groups is a *Lie group morphism* if it is smooth and a group homomorphism.

The smoothness condition ensures that the group structure is compatible with the manifold structure, allowing us to use tools from differential geometry to study the group. Some classic examples of Lie groups are finite groups, the space $\mathbb{R}^n$ under vector addition, $G = GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$, the general linear groups, $G = O(n)$ or $U(n)$, the orthogonal and unitary groups and $G = SO(n)$ or $SU(n)$, the special orthogonal and special unitary groups.

Definition 1.8. A *Lie subgroup* of a Lie group G is a subset $H \subset G$ which is a subgroup and also a submanifold (locally).

That is, H is both a subgroup and a smooth submanifold, making it a Lie group in its own right. We have the following classic characterisation, which we will not prove:

Theorem 1.9 (Cartan '30)**.** *Let $H \subseteq G$ be a subgroup of a Lie group. Then H is a Lie subgroup if and only if H is closed in G.*

Associated with every Lie group, we have the following important object:

Definition 1.10. Let G be a Lie group, and denote by left multiplication $L_g(h) \coloneqq g \cdot h$. A vector field $X \in \Gamma(TG)$ is called *left-invariant* if it is invariant when pushed forward by left translations, that is

$$(L_g)_*(X_p) = X_{L_g(p)}$$

for all $p, g \in G$. The collection of left-invariant vector fields is the *Lie algebra* of G, denoted by $\mathfrak{g}$.

The Lie algebra $\mathfrak{g}$ inherits an algebra structure, where the bracket on $\mathfrak{g}$ is induced by the Lie bracket of vector fields.

Lemma 1.11. *Every tangent vector at the identity extends uniquely to a left-invariant vector field. That is, the evaluation map*

$$\mathrm{ev} : \mathfrak{g} \to T_eG, \quad X \mapsto X(e)$$

is an isomorphism of vector spaces.

Thus, the dimension of the Lie algebra $\mathfrak{g}$ coincides with the dimension of the Lie group G.

Given a morphism of Lie groups $f : G \to H$, its differential at the identity induces a morphism of Lie algebras $f_* : \mathfrak{g} \to \mathfrak{h}$. In the case of the conjugation map, $c_g(h) \mapsto ghg^{-1}$, the associated pushforward map is known as the *adjoint map* of G, denoted by $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$.

Given a left-invariant vector field X, one can consider the associated family of diffeomorphisms φ_X^t. In particular, for each $X \in \mathfrak{g} \cong T_eG$, we can consider curve

$$\begin{aligned} \varphi_X : \mathbb{R} &\to G \\ t &\mapsto \varphi_X^t(e)\ . \end{aligned}$$

This is a one-parameter subgroup of G and a Lie group morphism from $(\mathbb{R}, +)$ to G. We have

Lemma 1.12. *For all $g \in G$, we have $\varphi_X^t(g) = g \cdot \varphi_X^t(e)$. In particular, every left-invariant vector field $X \in \mathfrak{g}$ is complete and defines a global flow $\varphi_X : \mathbb{R} \times G \to G$, $(t, g) \mapsto \varphi_X^t(g)$.*

Proof. Consider the curve $\gamma(t) = g \cdot \varphi_X^t(e)$. This curve satisfies $\gamma(0) = g$ and

$$\begin{aligned} \gamma'(0) &= \frac{d}{dt}\left(g \cdot \gamma(t)\right)\bigg|_{t=0} = (L_g)_* \left(\frac{d}{dt}\gamma(t)\bigg|_{t=0}\right) \\ &= (L_g)_*\, X = X_g\ . \end{aligned}$$

By the uniqueness of integral curves, the claim follows. ☐

Using this lemma, it follows that the *exponential map*

$$\begin{aligned} \exp : \mathfrak{g} &\to G \\ X &\mapsto \varphi_X^1(e) \end{aligned}$$

is always well defined. The exponential map is smooth, and its derivative at zero is the identity map $T_0\mathfrak{g} \cong \mathfrak{g} \to T_eG \cong \mathfrak{g}$. By the inverse function theorem, exp is a local diffeomorphism near $0 \in \mathfrak{g}$.

Our main interest in defining and studying Lie groups is that they are the natural group class to consider when considering actions on smooth manifolds. We define

Definition 1.13. Let M be a smooth manifold and G a Lie group. A *smooth left action* of G on M is a smooth map $\rho : G \times M \to M$ such that

(i) $\rho(e, p) = p$ for all $p \in M$,

(ii) $\rho(h, \rho(g, p)) = \rho(hg, p) \cdot p$ for all $p \in M$ and $g, h \in G$.

Given a left action ρ, there is an assocaited right action, given by $\tilde{\rho}(g, p) = \rho(g^{-1}, p)$.

Definition 1.14. Let $\rho : G \times M \to M$ be a smooth action. For each $X \in \mathfrak{g}$, we define the *fundamental vector field* $\tilde{X}$ on M by

$$X^*(p) = \frac{d}{dt}\bigg|_{t=0} \rho(\exp(tX), p).$$

The fundamental vector field is tangent to the orbit of the action and encodes the infinitesimal action of the Lie algebra.

Definition 1.15. An action of G on M is *free* if $g \cdot p \neq p$ for all $p \in M$ and all $g \in G$ with $g \neq e$.

Lemma 1.16. *Let $\rho : G \times M \to M$ be a free Lie group action. Then, the image of the orbit map $\mu_p : G \to M$, $g \mapsto g \cdot p$ is diffeomorphic to G for all $p \in M$.*

Proof. First, note that the orbit map μ_p is injective, since if $\mu_p(g) = \mu_p(h)$, $h^{-1}g$ fixes p, so $h^{-1}g = e$. Let us prove that the differential $d(\mu_p)_e : T_eG \to T_pM$ is injective. For $X \in T_eG$, consider $d(\mu_p)_e(X) = X^*(p)$ its fundamental vector field.

If $X \in \ker(d\mu_p)$, we get $X^*(p) = 0$, so the integral curve of X^* through p is constant: $\exp(tX) \cdot p = p \quad \forall t \in \mathbb{R}$. But this is equivalent to $\exp(tX) = e$ for all t, which implies $X = 0$, as needed. Therefore, the orbit map is an injective immersion, and therefore it's a diffeomorphism onto the image. $\square$

1.2 Lie algebras and representations

Let us now discuss the special case where M is a vector space and G acts by linear transformations.

Definition 1.17. A finitie *representation* of a Lie group G is a pair (V, π), with V a finite-dimensional vector space and $\rho : G \to GL(V)$ is a Lie group morphism.

A representation of a Lie algebra $\mathfrak{g}$ is a pair (V, ρ), where V is a vector space and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a Lie algebra morphism.

Although a representation is a pair (V, ρ), we will abbreviate it by either V or ρ whenever there is no ambiguity. Some examples of representations are the trivial and standard representations, and the adjoint representation $\mathrm{Ad}_g : G \to \mathfrak{g}$ that appeared earlier.

Definition 1.18. Let (V, ρ) and (W, τ) be two representations of G. A morphism of G-representations is a linear map $F : V \to W$ satisfying the G-equivarniance condition

$$F\big(\rho(g)\ v\big) = \tau(g) F(v)$$

for all $g \in G$ and $v \in V$. The collection of representation morphisms is denoted by $\mathrm{Hom}_G(V, W)$.

Similarly, one can define $\mathfrak{g}$-morphisms of representations of Lie algebras. The set of $\mathfrak{g}$-morphisms between V and W is denoted by $\mathrm{Hom}_{\mathfrak{g}}(V, W)$.

We have the following correspondence

Theorem 1.19. *Let G be a Lie group with Lie algebra $\mathfrak{g}$.*

(i) Every representation $\rho : G \to GL(V)$ of G defines a representation $\rho = d\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ of $\mathfrak{g}$. Moreover, any morphism between representations of G is automatically a morphism between the corresponding representations of $\mathfrak{g}$. Moreover, assume G is connected and simply connected.

(ii) Any representation of $\mathfrak{g}$ can be uniquely lifted to a representation of G.

(iii) We have an isomorphism $\mathrm{Hom}_G(V, W) \cong \mathrm{Hom}_{\mathfrak{g}}(V, W)$.

The key part of the proof is the existence of the lift of an algebra representation to a group representation. The idea of the construction is to use parallel transport, which we will introduce in Section 3. We postpone the proof of the theorem until then.

Let us discuss a key Lie group and Lie algebra representations. Recall the adjoint map $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$, given by the differential of the conjugation map at the identity. By varying the conjugation element g, one obtains a linear representation of G, known as the *adjoint representation*:

$$\begin{aligned} \mathrm{Ad} : G &\to \mathrm{End}(\mathfrak{g}) \\ g &\mapsto \mathrm{Ad}_g \ . \end{aligned}$$

Let us state two useful results about the role of the adjoint map in understanding the structure of Lie groups:

Lemma 1.20. *The differential of right multiplication is given by the adjoint map; $(R_g)_* = \mathrm{Ad}(g^{-1})$.*

Proof. Let $X \in \mathfrak{g}$ and φ_X^t its associated flow. Take $g \in G$. Then, by left invariance of X, we have

$$(R_g)_*(X) = (R_g)_*(L_{g^{-1}})_*(X) = (C_{g^{-1}})_*(X) = \mathrm{Ad}_{g^{-1}}(X) \ . \qquad \square$$

Lemma 1.21. *The differential $\mathrm{Ad}_* : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ coincides with the map $X \mapsto [X, \cdot]$ given by the bracket of vector fields.*

Proof. Exercise 6. □

A central question in Lie Theory is the classification of all the representations, up to isomorphism, of a given Lie group G. To try to answer that question, we need to introduce the following notions:

Definition 1.22. Let G be a Lie group and (V, ρ) a representation of G. A *subrepresentation* of (V, ρ) is a G-invariant linear subspace $W \subseteq V$ together with the restriction of ρ to W.

A representation is called *irreducible* if it has no subrepresentations other than 0 and V. Otherwise, it is called *reducible.*

Given a reducible representation, with a subrepresentation W, which gives us a short exact sequence

$$0 \to W \to V \to V/W \to 0$$

of representations. It is natural to ask whether this sequence splits, i.e. whether $V \cong W \oplus V/W$ as G-representations. If this were the case, we could see W and V/W as building blocks of the representation (V, ρ). Note that this makes understanding the representation easier, since the dimensions of the pieces are lower. In our case, we have

Theorem 1.23. *Let G be a compact Lie group. Then for any representation (V, ρ), there exist irreducible representations $(W_1, \rho_1), \ldots, (W_n, \rho_n)$ such that we have a decmoposition*

$$(V, \rho) \cong (W_1, \rho_1) \oplus \cdots \oplus (W_n, \rho_n)\ .$$

Proof. Recall from basic linear algebra that a standard way to choose a splitting of a short exact sequence of vector spaces is by choosing an inner product on V and taking its orthogonal complement as the natural representative of the quotient V/W. To extend this approach to the class of representations, we need to pick an inner product on V that is ρ-invariant.

Let us construct such an invariant inner product. Take $\langle \cdot, \cdot \rangle$ an arbitrary inner product on V, and consider new inner product

$$\langle v, w \rangle_G := \frac{1}{\mathrm{vol}(G)} \int_G \langle \rho(g)v, \rho(g)w \rangle dg$$

where dg is the invariant volume form of G, its Haar measuere.

Taking the orthogonal complement of W with respect to $\langle v, w \rangle_G$, we get the desired splitting of representations. ☐

It is useful to characterise the space of maps between two irreducible representations.

Theorem 1.24. *Let G be a Lie group, and consider (V, ρ_V) and (W, ρ_W) be two irreducible complex G-representations. Then*

$$\mathrm{Hom}_G(V, W) = \begin{cases} 0 & \textit{if } V \ncong W \\ \mathbb{C} & \textit{if } V \cong W \end{cases} .$$

Proof. Let $f : V \to W$ be a G-linear map. The idea of the proof is that every complex linear map (or an algebraic closed field) has an eigenvalue. But the corresponding eigenspace would be a subrepresentation of V. But since V is irreducible, that eigenspace must be equal to V, so $f = \lambda\,\mathrm{Id}$. But if $\lambda \neq \mathrm{Id}$, f induces an isomorphism of vector spaces. ☐

2 Bundles and more bundles

We now turn to the study of bundles. Our motivation comes from the fact that any manifold equipped with a Γ_G-structure carries an associated principal G-bundle, which we will define shortly. The existence of such a principal bundle can be used to construct topological obstructions to the existence of Γ_G structures. We begin by discussing general smooth fibre bundles, then specialise to principal bundles and vector bundles.

Definition 2.1. A *smooth fibre bundle* is a tuple (E, B, π, F), where E, B and F are smooth manifolds and $\pi : E \to B$ is a smooth surjective map between smooth manifolds which is locally trivial: for every $p \in B$, there exists an open neighborhood $U \subset B$ and a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ such that the following diagram commutes:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"] & U \times F \arrow[dl, "\mathrm{pr}_1"] \\
U &
\end{tikzcd} .$$

Here, B is called the *base*, E is the *total space*, and $F \cong \pi^{-1}(p)$ is the *fibre*. Some examples of fibre bundles are products $B \times F \to B$, $(p, f) \mapsto p$, and covering maps, $\pi : \widetilde{B} \to B$, with a discrete fibre. We will usually shorten the notation and denote a fibre bundle by $\pi : E \to B$, and omit to reference the fibre explicitly.

Definition 2.2. Let $\pi_1 : E_1 \to B_1$ and $\pi_2 : E_2 \to B_2$ be fibre bundles. A *bundle map* is a pair of smooth maps $\overline{f} : E_1 \to E_2$ and $f : B_1 \to B_2$ such that the diagram commutes:

$$\begin{tikzcd}
E_1 \arrow[r, "\overline{f}"] \arrow[d, "\pi_1"] & E_2 \arrow[d, "\pi_2"] \\
B_1 \arrow[r, "f"] & B_2
\end{tikzcd} .$$

When $B_1 = B_2 = B$ and $f = \mathrm{id}_B$, we call $\overline{f}$ a *bundle map* over B.

Definition 2.3. A *section* of a fibre bundle $\pi : E \to B$ is a smooth map $s : B \to E$ such that $\pi \circ s = \mathrm{id}_B$.

Given two local trivializations $\varphi_i : \pi^{-1}(U_i) \to U_i \times F$ and $\varphi_j : \pi^{-1}(U_j) \to U_j \times F$ with $U_i \cap U_j \neq \emptyset$, we can consider the transition map φ_{ij}, given by

$$\varphi_j \circ \varphi_i^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F, \quad (p, f) \mapsto (p, \alpha_{ij}(p)(f)),$$

where $\alpha_{ij} : U_i \cap U_j \to \mathrm{Diff}(F)$ is smooth. These transition functions satisfy the cocycle condition:

$$\alpha_{ii} = \mathrm{id}, \qquad \alpha_{ij} \circ \alpha_{jk} = \alpha_{ik} \quad \text{on} \quad U_i \cap U_j \cap U_k. \tag{1}$$

Conversely, given an open cover $\{U_i\}$ of B and smooth maps $\alpha_{ij} : U_i \cap U_j \to \mathrm{Diff}(F)$ satisfying the cocycle condition, we can reconstruct the fibre bundle up to isomorphism as follows: consider the space

$$\widehat{E} = \left(\coprod_i U_i \times F \right) / \sim,$$

where $(p, f) \in U_i \times F$ and $(q, g) \in U_j \times F$ are related if $p = q$ and $g = \alpha_{ij}(p)(f)$. Then, the tuple $(\widehat{E}, B, \widehat{\pi}, F)$ is a smooth fibre bundle, where the projection map is given simply by $\pi([p, f]) = p$. We leave it to the reader to verify the details that $\widehat{E}$ and E are isomorphic as fibre bundles.

This reconstruction shows that a fibre bundle is determined by its transition functions, up to a cocycle condition. In particular, it is not hard to prove:

Lemma 2.4. *Two collections of transition functions* $\{\phi_{ij}\}, \{\psi_{\alpha\beta}\}$*with respect to open covers* U_i *and* V_α *on* B*, with values on* $\mathrm{Diff}(F)$*, determine isomorphic fibre bundles if and only if there is a common refinement of the open covers over which the transition functions are compatible.*

Let us be rather vague on what we mean precisely by *compatible.* The right notion of compatibility using Čech cohomology is described in the addendum 2.1.1. From now on, we restrict our attention to fibre bundles with a ”small” structure group:

Definition 2.5. Let F be a smooth manifold and $G \subseteq \mathrm{Diff}(F)$ a Lie group acting smoothly on F on the left. A *smooth fibre bundle with fibre* F *and structure group* G is a smooth fibre bundle $\pi : E \to B$ with fibre F equipped with a family of local trivializations $\{(U_i, \varphi_i)\}$ such that the transition functions α_{ij} take values in G, i.e., $\alpha_{ij} \in \Gamma(U_i \cap U_j, G)$.

We say that the structure group can be *reduced* to a subgroup $H \subset G$ if there exists a system of local trivializations such that all transition functions take values in H. In particular, a fibre bundle is trivial (meaning isomorphic to the product) if its structure group reduces to the trivial group.

2.1 Principal G-bundles

Given a fibre bundle with structure group G, we can construct an associated fibre bundle $G \to P \to B$ as follows: Take a trivialising cover $\{U_i\}$ of E, with transition functions $\{\alpha_{ij}\} \in \Gamma(U_{ij}, G)$, since the fibre bundle has structure group G. Now, we may construct P using the same cover $\{U_i\}$ and transition functions $\{\alpha_{ij}\}$, where, rather than acting by diffeomorphisms, we act by left multiplication on the fibres G.

This construction motivates the definition

Definition 2.6. Given a Lie group G, a *principal* G*-bundle* is a smooth fibre bundle with fibre G and structure group G, where the action of the structure group is given by left-multiplication.

Note that not all G-bundles are necessarily principal G-bundles. In fact, they are characterised by the following proposition:

Proposition 2.7. *A principal G-bundle is equivalent to a smooth fiber bundle $\pi : P \to B$ together with a smooth right action $P \times G \to P$ such that:*

(i) The action is free and the orbits coincide with the fibres of π.

(ii) The local trivializations $\varphi : \pi^{-1}(U) \to U \times G$ are G-equivariant: $\varphi(p \cdot g) = (\pi(p), \sigma(p)g)$ for all $g \in G$.

Proof. Suppose $\pi : P \to B$ is a principal G-bundle. We need to define the right action of G on P. Using the trivialisation presentation $P = \left(\coprod_i U_i \times G\right) / \sim$, the group G acts on $\coprod_i U_i \times G$ on the right by right multiplication on G. The action descends to P since left and right multiplication commute, and satisfies the required conditions.

Conversely, assume we have a smooth bundle with a right free G-action satisfying conditions (i) and (ii). Consider two local trivializations

$$\phi : \pi^{-1}(U) \to U \times G \qquad p \mapsto (\pi(p), \sigma(p)) \, \psi : \pi^{-1}(V) \to V \times G \qquad p \mapsto (\pi(p), \tau(p)) \ ,$$

with $U \cap V \neq \emptyset$. In particular, we have

$$\begin{aligned} \psi \circ \phi^{-1} : (U \cap V) \times G \to& (U \cap V) \times G \\ (\pi(p), \sigma(p)) \mapsto& (\pi(p), \tau(p)) \\ &= \big((\pi(p), \big[\tau(p)\sigma(p)^{-1}\big]\sigma(p)\big) \ . \end{aligned}$$

We need to show that $\tau(p)\sigma(p)^{-1}$ is the required transition function. Indeed, acting by G on the right, we have

$$\begin{aligned} \psi \circ \phi^{-1}(pg) &= \big((\pi(p), \big[\tau(pg)\sigma(pg)^{-1}\big]\sigma(pg)\big) \\ &= \big((\pi(p), \big[\tau(p)gg^{-1}\sigma(p)^{-1}\big]\sigma(p)g\big) \\ &= \big((\pi(p), \big[\tau(p)\sigma(p)^{-1}\big]\sigma(p)g\big) \ , \end{aligned}$$

where we used G-equivariance in the second line. Thus, by condition (i), we see that there exists a unique group element for every point in the fibre that determines the transition map. Namely, the map

$$\begin{aligned} \alpha : U \cap V &\to G \\ \pi(p) &\mapsto \tau(p)\sigma(p)^{-1} \end{aligned}$$

is well-defined and smooth. ☐

Using this characterisation of principal G-bundles, we can define maps between principal bundles:

Definition 2.8. Let G, H be Lie groups, and $G \to P \to B$ and $H \to Q \to C$ principal bundles. A morphism of bundles is a tuple of maps $(\overline{f}, f, \widetilde{f})$, with $\overline{f} : P \to Q$, $f : B \to C$ smooth maps and $\widetilde{f} : G \to H$ a group morphism such that the diagram

$$\begin{tikzcd} P \arrow[r, "\overline{f}"] \arrow[d, "\pi"] & Q \arrow[d, "\pi"] \\ B \arrow[r, "f"] & C \end{tikzcd}$$

commutes, and such that

$$\overline{f}(p \cdot g) = \overline{f}(p) \cdot \widetilde{f}(g)$$

for all $g \in G$.

An isomorphism of principal bundles is a morphism of bundles with $\widetilde{f} = \mathrm{Id}_G$, such that f and $\overline{f}$ are diffeomorphisms.

We have a very clean characterisation of when a principal G-bundle is trivial:

Lemma 2.9. *A principal G-bundle over a connected base B is trivial if and only if it has a section.*

Proof. If $P = B \times G$, then $s : B \to P$, given by $s(b) = (b, e)$ is a section of P.

Conversely, let $P \to B$ be a principal G-bundle with a section $s : B \to P$. Then, the map

$$\begin{aligned} \overline{f} : B \times G &\to P \\ (b, g) &\mapsto s(b)g \end{aligned}$$

is a principal G-bundle. □

We have constructed a principal G-bundle P out of a fibre bundle E with structure group G. Let us now go the other way.

Definition 2.10. Let $G \to P \xrightarrow{\pi_P} B$ be a principal G-bundle, F smooth manfiold and $\rho : G \to \mathrm{Diff}(F)$ a group morphism. The *associated bundle* $E_\rho := P \times_\rho F$ is the quotient $(P \times F)/G$, where G acts on the right as

$$\begin{aligned} (P \times F) \times G &\to P \times F \\ \big((p, f), g\big) &\mapsto (p \cdot g, \rho(g^{-1}) \cdot f) \ . \end{aligned}$$

The projection map $\pi_E : E \to B$ is given by $\pi_E\big([(p, f)]\big) = \pi_P(p)$.

Note that the projection map π_E is well-defined, precisely because the G-orbits in P coincide with the fibres of π_P. With the preceding discussion, we have essentially proved

Theorem 2.11. *Let $F \to E \xrightarrow{\pi} B$ be a fibre bundle with structure group G, and let $P \xrightarrow{\pi_P} B$ be the corresponding principal G-bundle constructed from the transition functions of E. Consider the natural action $\rho : G \to \mathrm{Diff}(F)$ given by the structure group representation. Then the associated bundle $E_\rho = P \times_\rho F$ is isomorphic to E.*

That is, every fibre bundle with structure group G can be realised as an associated bundle to a principal G-bundle. The reader is invited to fill in the gaps in the proof.

Given a principal G-bundle P, one can ask whether its structure group is a proper subgroup of G. That is, if we can find local trivializations of P such that all transition maps take values in a Lie subgroup $H \subset G$. In this case, there exists a principal H-bundle Q, and a morphism of principal bundles

$$\begin{tikzcd} P \arrow[rr, "\iota"] \arrow[dr, "\pi"] & & Q \arrow[dl, "\pi"] \\ & B & \end{tikzcd} ,$$

where the group morphism $\widetilde{\iota} \colon H \to G$ is the inclusion. We are interested in characterising when a principal G-bundle can be *reduced* to a principal H-bundle, as described. To do so, let us introduce the following associated fibre bundle:

Proposition 2.12. *Let P be a principal G-bundle, $H \subset G$ a closed subgroup. The quotient P/H, given by the restriction of the right action of G to H, is a fibre bundle with fibre G/H. In particular,*

$$P/H \cong P \times_{\mathrm{Id}} \big(G/H\big) \ .$$

Proof. Let us construct a fibre bundle map Φ between the two spaces. We set

$$\begin{aligned} \Phi : P/H &\to P \times_{\mathrm{Id}} \big(G/H\big) \\ [p] &\mapsto [(p, eH)] \ . \end{aligned}$$

First, note that Φ is well defined. Indeed, the image of a different representative of the class, given by ph for $h \in H$, is mapped to

$$[(ph, eH)] = [(p, h^{-1}H)] = [(p, eH)] \ .$$

Moreover, we have the commutative diagram

$$\begin{tikzcd} P/H \arrow[rr, "\Phi"] \arrow[dr, "\pi"] & & P \times_{\mathrm{Id}} \big(G/H\big) \arrow[dl, "\pi'"] \\ & B & \end{tikzcd} ,$$

which shows Φ is a bundle morphism. □

We refer to P/H as the *quotient bundle*. Note that the quotient bundle is not a principal bundle anymore (unless H is trivial). We have

Proposition 2.13. *A principal G-bundle P admits a reduction to an H-bundle if and only if the associated quotient bundle P/H admits a section.*

Proof. Assume the principal bundle $G \to P \xrightarrow{\pi_P} B$ admits a reduction to the principal H-bundle $H \to Q \xrightarrow{\pi_Q} B$. For each $b \in B$, the fiber Q_b is an H-orbit inside P_b. Define the map

$$\begin{gathered} s : B \longrightarrow P/H \\ b \mapsto [q] \in P_b/H \end{gathered}$$

[for any $b \in Q_b$. This is well-defined because all points in Q_b lie in the same H-orbit, hence define the same element of P_b/H. Smoothness follows from the fact that Q is a smooth subbundle. Clearly $\pi \circ s = \mathrm{id}_M$, so s is a global section of P/H.

Conversely, let $s : B \to P/H$ be a section. Consider the pre-image of $s(B)$ under the projection map $\pi : P \to P/H$. This is a principal H-bundle Q sitting inside P. More specifically, consider the pullback diagram:

$$\begin{tikzcd} s^*P \arrow[r] \arrow[d] & P \arrow[d] \\ B \arrow[r, "s"] & P/H \end{tikzcd}$$

Here $P \to P/H$ is the canonical projection, which is a principal H-bundle because the right action of H on P is free and H acts transitively on the fibres of this projection.

The pullback s^*P is a principal H-bundle over B (cf. Exercise 7). The natural inclusion $s^*P \hookrightarrow B \times P \to P$ given by projection onto the second factor is an H-equivariant embedding, whose image is exactly the preimage of the section s. This image is a smooth submanifold of P and, since the H-action is free and the map to M is surjective, it defines the required principal H-subbundle $Q \subset P$. □

Note that the case $H = \{e\}$ was proved directly in Lemma 2.9.

2.1.1 Addendum: Classification of principal G-bundles

We include a brief discussion on the classification problem of principal G-bundles. The problem is nowadays well-understood (in abstract terms), and we have power algebro-topological techniques to study the problem. The one we will be interested in today is Čech cohomology. We will not go into detail about the constructions, nor will we provide the technical details.

Given an open cover $\mathcal{U} := \{U_i\}_{i \in I}$ and a Lie group G, we consider the map S_G that assigns to an open subset $U \subseteq M$, the smooth functions $\phi_i : U \to G$, denoted by $S_G(U)$ [1] . We define the *Čech*

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[1] For the expert reader, S_G is, more generally, the sheaf of smooth functions into G. Alternatively, one might take the sheaf of continuous functions.

groups for the cover $\{U_i\}$:

$$\begin{aligned}\check{\mathcal{C}}^0(S_G,\mathcal{U}) &\coloneqq \prod_{i\in I} S_G(U_i)\ ,\\ \check{\mathcal{C}}^1(S_G,\mathcal{U}) &\coloneqq \prod_{i\neq j} S_G(U_i\cap U_j)\ ,\\ &\dots\\ \check{\mathcal{C}}^k(S_G,\mathcal{U}) &\coloneqq \prod_{\substack{i_1,\dots,i_k\\ \text{pairwise distinct}}} S_G(U_{i_1}\cap\dots\cap U_{i_k})\ ,\end{aligned}$$

where the products are taken in a formal sense. We define the *Čech coboundry operator* $\delta : \check{\mathcal{C}}^p(S_G,\mathcal{U}) \to \check{\mathcal{C}}^{p+1}(S_G,\mathcal{U})$, given by

$$(\delta\sigma)_{a_0\dots a_{p+1}} \coloneqq \prod_{j=0}^{p+1} \sigma^{(-1)^j}_{a_0\dots\hat{a}_j\dots a_{p+1}}\big|_{U_{a_0}\cap\dots\cap U_{a_{p+1}}}\ .$$

A (tedious) computation shows that $\delta^2=0$, so $\big(\check{\mathcal{C}}^q(S_G,\mathcal{U}),\delta\big)$ is a complex of sheaves. In particular, we can define the (relative) Čech cohomology groups:

$$\check{H}^q(S_G,\mathcal{U}) \coloneqq \frac{\ker\big(\delta:\mathcal{C}^q\to\mathcal{C}^{q+1}\big)}{\operatorname{im}\big(\delta:\mathcal{C}^{q-1}\to\mathcal{C}^q\big)}\ .$$

In degree zero, we have $\check{H}^0=\ker\delta$, with $\delta(s)_{ij}=s_j|_{U_i\cap U_j}\cdot s_i^{-1}|_{U_i\cap U_j}$. Hence, we see $\check{H}^0(S_G,\mathcal{U})=\Gamma(M,G)$, globally defined functions from M to G. Note that, in this case, there is no dependence of the chosen cover $\mathcal{U}$. Unfortunately, this is not true for the higher Čech cohomology groups.

To suppress the dependence on the cover $\mathcal{U}$, note that, for a finer cover $\mathcal{V}\supseteq\mathcal{U}$, we have injective maps $\check{\mathcal{C}}^p(S_G,\mathcal{U})\to\check{\mathcal{C}}^p(S_G,\mathcal{V})$, which induce maps (not necessarily injective) in cohomology. In particular, one can take the *direct limit* over all covers, and define

$$\check{H}^p(S_G) \coloneqq \varinjlim_{\mathcal{U}}\ \check{H}^p(S_G,\mathcal{U})\ .$$

In our context of principal bundles, we have

Theorem 2.14. *Let M be a smooth manifold, G a Lie group. Then there is a one-to-one correspondence between principal G-bundles on M and $\check{H}^1(S_G)$.*

Proof. Let us sketch the proof for a given open cover $\mathcal{U}$. If $P\to M$ is a principal G-bundle that trivialises over the open cover $\mathcal{U}$, the cocycle conditions (1) guarantee that we can associate to it an element $\check{H}^1(S_G,\mathcal{U})$.

Conversely, given a cocycle $[\Phi_{\alpha\beta}]\in\check{H}^1(S_G,\mathcal{U})$, one can build a principal G-bundle P using $\Phi_{\alpha\beta}$ as transition functions. Note that this is well-defined (and hence we have a one-to-one map), since if we picked a different cocycle representative, the difference would be a coboundary term δf_a, which would not modify the principal G-bundle, as it simply gives rise to a compatible diffeomorphism of the fibre. ☐

Understanding the set $H^1(S_G)$ for a given manifold M is not a trivial endeavour. However, there are two cases of relevance in which we have a complete understanding of $H^1(S_G)$.

Theorem 2.15. *Let M be a closed manifold.*

(i) Let $G = \mathbb{R}^* \cong \mathrm{GL}(1,\mathbb{R})$. Then $H^1(S_G)$ has a natural (abelian) group structure, and is diffeomorphic to $H^1(M,\mathbb{Z}/2\mathbb{Z})$.

(ii) Let $G = \mathbb{C}^* \cong \mathrm{GL}(1,\mathbb{C})$. Then $H^1(S_G)$ has a natural (abelian) group structure, and is diffeomorphic to $H^2(M,\mathbb{Z})$.

The proof idea is that the short exact sequences of groups

$$0 \to \mathbb{R} \xrightarrow{\exp} \mathbb{R}^* \to \mathbb{Z}/2\mathbb{Z} \to 0 \ , \tag{2a}$$

$$0 \to \mathbb{Z} \to \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \to 0 \tag{2b}$$

induce long exact sequences of Čech cohomologies. The claim then follows by proving some cohomological vanishing and identifying certain Čech cohomology groups with the simplicial cohomology groups. We omit the details.

2.2 Frame bundles and G-structures

Let us now focus on a specific class of fibre bundles with finite structure group: vector bundles.

Definition 2.16. Let K be a field. A fibre bundle (E, B, π, K^n) is called a *vector bundle* if the structure group G of the fibre bundle is a subgroup of $\mathrm{GL}(n, K)$.

The associated principal $\mathrm{GL}(n, K)$-bundle is called the *frame bundle* of E, and is denoted by $\mathrm{Fr}(E)$.

In our case, we only consider $K = \mathbb{R}$ or $K = \mathbb{C}$, and refer to them as real and complex vector bundles, respectively.

In the case where the vector bundle $E = TM$ is the tangent bundle of M, we abbreviate its frame bundle by $\mathrm{Fr}(M)$. Note that the frame bundle of E encodes the information of all the associated vector bundles to E via different representations of $\mathrm{GL}(n, K)$, such as its dual E^*, exterior and symmetric powers, $\Lambda^k E$ and $\mathrm{Sym}^k E$, etcetera.

Let us now turn back to our motivating problem and describe structures on manifolds from the point of view of principal bundles.

Definition 2.17. Let M be a manifold of dimension n, and $G \subset \mathrm{GL}(n, R)$ a closed subgroup. A G-structure on M is a reduction of the frame bundle $\mathrm{Fr}(M)$ to a principal G-bundle.

In view of Definition 1.3, it is clear that

Lemma 2.18. *A Γ_G-manifold M carries a natural associated G-structure.*

The converse is, in general, not true. When a G-structure is compatible with a Γ_G-structure, it is called *integrable.* We will study the problem of when a G-strucutre is integrable in Section 4.

Let us discuss how a geometric structure defines a G-structure on M, and vice-versa. Let T_0 be an element of the tensor algebra $\mathcal{T}(\mathbb{R}^n)$ over $\mathbb{R}^n$ and set $G = \mathrm{Stab}(T_0) \subseteq \mathrm{GL}(n, \mathbb{R})$ the stabiliser of T_0, which is a Lie subgroup of the general linear group. Since a point $u \in \mathrm{Fr}(M)$ can be understood as a vector space isomorphism $u : \mathbb{R}^n \to T_{\pi(u)}M$, we obtain an induced isomorphism $u_* : \mathcal{T}(\mathbb{R}^n) \to \mathcal{T}(T_{\pi(u)}M)$. The invariance of T_0 under G implies that the tensor field $T := u_*(T_0)$ defines a local section of the associated bundle $\mathrm{Fr}(M)/G$.

Therefore, Proposition 2.13 gives us an equivalence between admitting a G-structure and admitting a G-invariant tensor field T. In this case, we say that the G-structure is defined by the tensor T_0. Similarly, given a collection of tensors $T_0, \ldots, T_k$, we have a G-strucutre, with $G = \bigcap_{i=0}^k \mathrm{Stab}(T_i)$. This motivates the following

Definition 2.19. A group $G \subseteq \mathrm{GL}(n, \mathbb{R})$ is called *algebraic* if there exists a (finite) collection $\{T_0, \ldots, T_k\} \subseteq \mathcal{T}(\mathbb{R}^n)$ such that G is the stabiliser of such collection.

It is worth noting that the standard definition of algebraic groups typically found in the literature differs from the one given here. It is a non-trivial result of Chevalley (and Hilbert) that the two are tightly related. However, we will make no use of the algebraic characterisation of Chevalley–Hilbert in our discussion, so we do not go into further detail. Throughout these notes, we will focus on G-structures where G is an algebraic group.

Let us discuss some examples of G-structures. Throughout, we consider $V \cong \mathbb{R}^n$, the standard real vector space. First, let us briefly discuss some notational ambiguity with respect to the literature and historical terminology. We are treating G-structures as algebraic objects, and reserve the term integrable for those that admit a compatible Γ_G-structure. Historically, however, the choice of a G-structure was referred to as an *almost* G-structure, while the term G-structure was reserved for the integrable ones[2]. The most notable example of this historical legacy arises in complex geometry, where one talks about almost complex and complex structures.

Example 2.20 ($\mathrm{SL}(n, \mathbb{R})$-strucutres)**.** *The special linear group is the stabiliser of a non-zero volume element* $v \in \Lambda^n(\mathbb{R}^n)^*$. *The associated obstruction bundle* $\mathrm{Fr}(M)/\mathrm{SL}(n, \mathbb{R})$ *has fibre* $\mathbb{R} \setminus \{0\}$. *In particular, it follows that the bundle has a section if the line bundle* $\Lambda^n T^*M$ *is trivial.*

Example 2.21 ($\mathrm{GL}(n, \mathbb{C})$-strucutres)**.** *The complex general linear group* $\mathrm{GL}(n, \mathbb{C})$ *is the stabiliser of an endomorphism* $J \in \mathrm{End}(2n, \mathbb{R})$, *squaring to minus the identity* $J^2 = -\mathrm{Id}_{2n}$. *As usual, we can identify* $\mathbb{R}^{2n}$ *with the complex vector space* $\mathbb{C}^n$ *via the map*

$$(a + ib)v := a + b(Jv) \ .$$

The canonical complex structure J_0 *on* $\mathbb{R}^{2n}$ *is given in block form with respect to the standard basis*

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[2] The notational confusion becomes even worse when introducing weak integrability conditions, as we shall see later on.

by

$$J_0 = \begin{pmatrix} \mathrm{Id}_n & 0 \\ 0 & \mathrm{Id}_n \end{pmatrix} ,$$

and we can identify $\mathrm{GL}(n,\mathbb{C})$ *with the subgroup of matrices that commute with it.*

By Proposition 2.13, the existence of a $\mathrm{GL}(n,\mathbb{C})$*-strucutre on* M*, corresponds to a section of the fibre bundle* $\mathrm{Fr}(M)/\mathrm{GL}(n,\mathbb{C})$*, which can be shown to be a* $\mathrm{O}(2n)/\mathrm{U}(n)$*-bundle. It is generally a deep topological question whether this bundle admits a section or not. The problem is well-understood from the point of view of classifying spaces and obstruction theory (cf. [MS74, §12]), and it allows us to phrase necessary and sufficient conditions for the existence of a*$\mathrm{GL}(n,\mathbb{C})$*-structure in terms of very explicit topological conditions in low dimensions:*

(i) For $n = 1$*,* M *admits an almost complex structure if and only if* M *is orientable (equiv.* $w_1(M) = 0$*).*

(ii) For $n = 2$*,* M *admits an almost complex structure if and only if* M *is orientable and there exists* $h \in H^2(M,\mathbb{Z})$ *such that*

$$h^2 = 3\sigma(X) + 2\chi(X) \qquad\qquad h \equiv_2 w_2(X) .$$

Example 2.22 ($\mathrm{Sp}(2n,\mathbb{R})$-strucutres)**.** *The symplectic group* $\mathrm{Sp}(2n,\mathbb{R})$ *is defined as the stabiliser of the canonical symplectic structure on* $\mathbb{R}^{2n}$*:*

$$\omega = e^1 \wedge e^{n+1} + e^2 \wedge e^{n+2} + \cdots + e^n \wedge e^{2n} ,$$

where $\{e^1, e^2, \ldots, e^{2n}\}$ *is the canonical basis of* $\mathbb{R}^{2n}$*. Using the canonical complex structure* J_0*, the group* $\mathrm{Sp}(2n,\mathbb{R})$ *can be identified with*

$$\mathrm{Sp}(n,\mathbb{R}) = \{A \in \mathrm{GL}(2n,\mathbb{R}) | A^t J_0 A = J_0\} .$$

Example 2.23 ($\mathrm{GL}(n,\mathbb{H})\,\mathrm{Sp}(1)$-structures)**.** *Let* $\mathrm{GL}(n,\mathbb{H}) \subseteq \mathrm{GL}(4n,\mathbb{R})$ *the subgroup group of non-singular quaternionic* $n \times n$ *matrices under the identification* $\mathbb{R}^{4n} \cong \mathbb{H}$*, and consider the product subgroup* $\mathrm{GL}(n,\mathbb{H}) \times \mathrm{GL}(1,\mathbb{H}) \subseteq \mathrm{GL}(4n,\mathbb{R})$*. We define the group* $\mathrm{GL}(n,\mathbb{H})\,\mathrm{Sp}(1)$ *as the quotient* $\mathrm{GL}(n,\mathbb{H}) \times \mathrm{GL}(1,\mathbb{H})/\mathbb{R}^*$*, where* $\mathbb{R}^*$ *is the centre of the group.*

A reduction to a $\mathrm{GL}(n,\mathbb{H})\,\mathrm{Sp}(1)$*-structure corresponds to the existence of a local trivialisation* $\{I_1, I_2, I_23\}$ *of a rank 3 subbundle* $\mathcal{Q} \subseteq \mathrm{End}(TM)$ *satisfying the quaternion algebra relations:*

$$I_1^2 = I_2^2 = I_3^2 = -\,\mathrm{Id}_{4n} \qquad\qquad I_1 \circ I_2 = I_3 .$$

Example 2.24 ($\mathrm{GL}(n,\mathbb{H})$-structures)**.** *A* $\mathrm{GL}(n,\mathbb{H})$*-structure is a special case of the preceding case, where the local sections* I_1, I_2, I_3 *are globally defined, so the endomorphism bundle* $\mathrm{End}(TM)$ *contains a rank three trivial subbundle.*

Example 2.25 ($\mathrm{GL}(p,n;\mathbb{R})$-strucutres)**.** *For* $1 \leq p < n$*, we consider the subgroup of the general linear group given by*

$$\mathrm{GL}(p,n;\mathbb{R}) := \left\{ \begin{pmatrix} A & 0 \\ * & B \end{pmatrix} \;\middle|\; A \in \mathrm{GL}(p,\mathbb{R}) \ , \ B \in \mathrm{GL}(n-p,\mathbb{R}) \right\} .$$

The group $G = \mathrm{GL}(p, n; \mathbb{R})$ *stabilises a* p*-dimensional distribution* $\mathcal{D}$ *of the tangent bundle* TM*. Alternatively, the* p*-dimensional distribution* $\mathcal{D}$ *can be encoded in a projector* $P \in \mathrm{GL}(TM)$*,* $P^2 = P$ *of rank* p*.*

Example 2.26 ($\mathrm{GL}(p, \mathbb{R}) \times \mathrm{GL}(n-p, \mathbb{R})$-structures)**.** *As a particular case of the previous instance, the group* $G = \mathrm{GL}(p, \mathbb{R}) \times \mathrm{GL}(n-p, \mathbb{R}) \subseteq \mathrm{GL}(n, \mathbb{R})$ *stabilises a splitting of the tangent bundle* $TM = \mathcal{D} \oplus \widetilde{\mathcal{D}}$*, with* $\mathcal{D}$ *a* p*-dimensional distribution. Equivalently, this corresponds to having an endomorphism* $P \in \mathrm{End}(TM)$ *satisfying*$P^2 = \mathrm{Id}$*. Given* P*, we can construct two projectors* $P_\pm \coloneqq \frac{1}{2}(\mathrm{Id} - P)$ *satisfying the conditions*

$$[P_+, P_-] = 0 \qquad\qquad P_+ + P_- = \mathrm{Id}_n \ .$$

The two distributions correspond to the kernel of the two projector operators.

Example 2.27. $\mathrm{O}(n)$*-strucutres The orthogonal group* $\mathrm{O}(n, \mathbb{R}) = \{A \in \mathrm{GL}(n, \mathbb{R}) | A^t A = \mathrm{Id}_n\}$ *is the stabiliser of the standard Euclindean inner product in* $\mathbb{R}^n$*. Therefore, the bundle* $\mathrm{Fr}(M)/O(n)$ *can be identified with the subbundle of* $\mathrm{Sym}^2 T^*M$*, of positive definite symmetric bilinear forms.*

Using a partition of unity (subordinate to a trivialisation of $\mathrm{Fr}(M)$*), one sees that any manifold* M^n *admits an* $\mathrm{O}(n)$*-structure.*

More generally, we have the following general result from Lie group theory:

Theorem 2.28 (Cartan-Iwasawa)**.** *Every Lie group* G *admits a maximal compact Lie subgroup* K*. The quotient* G/K *is homeomorphic to* $\mathbb{R}^m$ *for some* m*.*

In particular, it follows that every G-structure admits a reduction to a K-structure, since any $\mathbb{R}^n$ bundle always admits a section, namely the zero section. Alternatively, one might argue using partitions of unity.

Let us revise the previous cases when they are reduced to their maximal compact subgroup:

Example 2.29 ($\mathrm{SO}(n)$-structure)**.** *We have* $\mathrm{SO}(n) \cong \mathrm{O}(n) \cap \mathrm{SL}(n, \mathbb{R})$ *is the maximal compact subgroup of the special linear group.*

Example 2.30 ($\mathrm{U}(n)$ and $\mathrm{SU}(n)$-strucutres)**.** *The unitary group* $\mathrm{U}(n) = \mathrm{O}(2n) \cap \mathrm{Sp}(2n, \mathbb{R}) \cong \mathrm{O}(2n) \cap \mathrm{GL}(n, \mathbb{C})$ *is the maximal subgroup of both the symplectic and complex general linear groups. Indeed, notice that, given a Riemannian metric* g *and almost complex structure* J*,* $\omega \coloneqq g(\cdot, J\cdot)$ *defines a symplectic strucutre on* M*.*

Similarly, the special unitary group $\mathrm{SU}(n) = \mathrm{O}(n) \cong \mathrm{SL}(n, \mathbb{C})$ *is the maximal subgroup of the complex special linear group. It is the stabiliser of the pair of forms:*

Example 2.31 ($\mathrm{Sp}(n)\,\mathrm{Sp}(1)$)**.** *The group* $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$ *is the maximal compact subgroup of* $\mathrm{GL}(n, \mathbb{H})\,\mathrm{Sp}(1)$*. Moreover, for* $n > 1$*, it is precisely the stabiliser of the* 4*-form* $\Omega \in \Lambda^4(\mathbb{R}^{4n})^*$ *known as the Cayley* *form. In normal coordinates $\{x_1, y_1, z_1, w_1, \ldots, x_n, y_n, z_n, w_n\}$, consider the three canonical symplectic structures*

$$\omega_1 = \sum_{i=1}^{n} x^i \wedge y^i + z^i \wedge w^i \qquad \omega_2 = \sum_{i=1}^{n} x^i \wedge z^i - y^i \wedge w^i$$

$$\omega_3 = \sum_{i=1}^{n} x^i \wedge w^i + y^i \wedge z^i \ .$$

Then the Cayley form Ω is given explicitly as

$$\Omega = \frac{\omega_1^2 + \omega_2^2 + \omega_2^3}{3} \omega$$

Example 2.32 (Sp(n))**.** *The group* Sp(n) *is the maximal compact subgroup of* GL($n, \mathbb{H}$) *and* Sp($2n, \mathbb{C}$)*. Thus, it can be understood as either the stabiliser of a Riemannian metric g compatible with a quaternionic triple I_1, I_2, I_3, or a hermitian metric h compatible with a complex symplectic form ω_c.*

As previously mentioned, we would like to understand when a G-structure is integrable. For now, we prove that all SL($n, \mathbb{R}$)-structures on a manifold are integrable, using the Moseer trick:

Lemma 2.33 (Moser trick)**.** *Let $\{\alpha_t\}_{t\in[0,1]} \subseteq \Omega^k(M)$ be a smooth family of forms, and $\{X_t\}_{t\in[0,1]} \subseteq \mathfrak{X}(M)$ family of compactly supported vector fields solving the ODE*

$$\partial_t \alpha_t + \mathcal{L}_{X_t} \alpha_t = 0 \ . \tag{3}$$

Then there exists a family of diffeomorphisms $\{\phi_t\}_{t\in[0,1]} \subseteq \mathrm{Diff}(M)$ such that $\phi_t^* \alpha_t = \alpha_0$.

Proof. Let ϕ_t be the flow associated to X_t, so $\partial_t \psi_t = X_t$. Then, by the chain rule and the assumption, Eq. (3), we have

$$\partial_t(\phi_t^* \alpha_t) = \mathcal{L}_{X_t} (\phi_t^* \alpha_t) + \phi_t^*(\partial_t \alpha_t) = \phi_t^* \big(\partial_t \alpha_t + \mathcal{L}_{X_t} \alpha_t\big) = 0 \ . \qquad \square$$

Using this lemma, it follows that

Corollary 2.34. *Every* SL($n, \mathbb{R}$)*-structure is integrable.*

Proof. Let U be a contractible neighbourhood of M, with coordinates $x_1, \ldots, x_n$, and $\alpha_0 \in \Omega^n(M)$ a local volume form coming from the SL($n, \mathbb{R}$)-structure, compatible with the orientation of the coordinates $x_1, \ldots, x_n$.

The form $(dx_1 \wedge \ldots \wedge dx_n) - \alpha_0$ is closed, and by the Poincaré Lemma, exact: there is an $(n-1)$-form β such that $d\beta = (dx_1 \wedge \ldots \wedge dx_n) - \alpha_0$.

We claim the family of volume forms $\alpha_t = \alpha_0 + t d\beta$ satisfies the conditions of the Moser trick. Indeed, let X_t be the family of vector fields given by the equation $\beta + X_t \lrcorner \alpha_t = 0$, whose existence follows from linear algebra. Then

$$\partial_t \alpha_t + \mathcal{L}_{X_t} \alpha_t = d\beta + X_t \lrcorner d\alpha_t + d(X_t \lrcorner \alpha_t) = 0 \ ,$$

as needed. The pullback of the coordinates under ϕ_1 give the desired $\Gamma_{\mathrm{SL}(n,\mathbb{R})}$-structure. $\square$

2.3 Spin strucutres

There is another class of geometrically relevant structures that are usually included in the world of G-structures, although they are slightly different in nature from the ones discussed above. Rather than trying to find a smaller and smaller subgroup $G \subseteq \mathrm{GL}(n, \mathbb{R})$ for which the frame bundle can be reduced to a principal G-bundle, we are trying to "lift" the frame bundle to a larger principal bundle that covers the frame bundle. Let us give the precise definition:

Definition 2.35. Let G be a Lie group, and

$$1 \to N \to \widetilde{G} \xrightarrow{\rho} G \to 1$$

a group extension of G by N. Given a manifold M, equipped with a G-strucutre $\pi : P \to M$, we say the G-structure can be *enriched* to a $\widetilde{G}$-structure if there exists a principal $\widetilde{G}$-bundle Q and a principal bundle morphism $\Phi : Q \to P$ such that the induced group morphism $\widetilde{\Phi}$ is precisely ρ.

Given G, there are a priori many enriched G-structures one may consider. For instance, if G is not simply connected, one can consider extensions of G by subgroups of its fundamental group $\pi_1(G)$. A prominent example of this is the metaplectic group

$$1 \to \mathbb{Z}_2 \to \mathrm{MSp(2n}, \mathbb{R}) \to \mathrm{Sp}(2n, \mathbb{R}) \to 1 \ ,$$

the double cover of the symplectic group, which plays a prominent role in symplectic geometry and mathematical physics.

For this course, we will focus our attention on the generalised Spin groups:

$$1 \to \mathbb{Z}_2 \to \ \mathrm{Spin}(n) \to \mathrm{SO}(n) \to 1 \ , \tag{4a}$$

$$1 \to \mathrm{U}(1) \to \ \mathrm{Spin}^c(n) \to \mathrm{SO}(n) \to 1 \ , \tag{4b}$$

$$1 \to \mathrm{Sp}(1) \to \ \mathrm{Spin}^h(n) \to \mathrm{SO}(n) \to 1 \ . \tag{4c}$$

The $\mathrm{Spin}(n)$ group is simply the universal cover of $\mathrm{SO}(n)$ for $n \geq 3$, with $\mathrm{Spin}(2) \to \mathrm{SO}(2)$ the circle double-cover. The $\mathrm{Spin}^c(n)$ and $\mathrm{Spin}^h(n)$ are best described in terms of the Spin groups:

$$\mathrm{Spin}^c(n) \cong \mathrm{Spin}(n) \times \mathrm{U}(1)/\sim_\sigma \qquad \text{and} \qquad \mathrm{Spin}^h(n) \cong \mathrm{Spin}(n) \times \mathrm{Sp}(1)/\sim_\sigma \ ,$$

where we quotient by the involution $\sigma : (\mathrm{Id}, \mathrm{Id}) \mapsto (-\mathrm{Id}, -\mathrm{Id})$. We have the following results for the existence of Spin and Spin^c-structures.

Proposition 2.36. *Let M^n be an oriented n-dimensional smooth connected manifold, so it admits an $\mathrm{SO}(n)$-structure. Then M admits a $\mathrm{Spin}(n)$-structure if and only if the second Stiefel-Whitney class $w_2(M) \in H^2(M, \mathbb{Z}_2)$ vanishes. In that case, the space of $\mathrm{Spin}(n)$-structures on M is a torsor over $H^1(M, \mathbb{Z}_2)$.*

Proposition 2.37. *Let M^n be an oriented n-dimensional smooth connected manifold. Then M admits a $\mathrm{Spin}^c(n)$-structure if and only if the second Stiefel-Whitney class $w_2(M) \in H^2(M, \mathbb{Z}_2)$ is the reduction of an integral class $H \in H^2(M, \mathbb{Z})$. If that case, the space of $\mathrm{Spin}^c(n)$-structures on M is a torsor over $H^2(M, \mathbb{Z})$.*

Let us give a brief outline of the proof using Cech cohomology for the *Spin* case. Fix an $\mathrm{SO}(n)$-structure on M (i.e. a metric), which corresponds to an element $\alpha \in \check{H}^1(S_{\mathrm{SO}(n)})$. A Spin structure corresponds to a lift of α to a class $\hat{\alpha} \in \check{H}^1(S_{\mathrm{Spin}(n)})$.

Now, the short exact sequence (4a) yields a long exact sequence of cohomology

$$\cdots \to H^1(M, \mathbb{Z}/2\mathbb{Z})) \to \check{H}^1(S_{\mathrm{Spin}(n)}) \xrightarrow{\phi} \check{H}^1(S_{\mathrm{SO}(n)}) \to H^2(M, \mathbb{Z}/2\mathbb{Z})) \to \dots ,$$

where we are identifying $H^i(M, \mathbb{Z}/2\mathbb{Z})) \cong \check{H}^i(S_{\mathbb{Z}/2\mathbb{Z}})$ without proof. Hence, the obstruction to lift α to $\hat{\alpha}$ is given by an element in the cokernel of ϕ, $H^2(M, \mathbb{Z}/2\mathbb{Z}))$, which we denote by $w_2(M)$ and refer to as the second Stiefel-Whitney class[3]. Similarly, one proves that the space of $\mathrm{Spin}(n)$-structures is parametrised by $H^1(M, \mathbb{Z}/2\mathbb{Z})$.

For the $\mathrm{Spin}^c(n)$ case, one has the long exact sequence

$$\cdots \to \check{H}^1(M, S_{\mathrm{U}(1)}) \to \check{H}^1(S_{\mathrm{Spin}^c(n)}) \xrightarrow{\phi} \check{H}^1(S_{\mathrm{SO}(n)}) \to \check{H}^2(M, S_{\mathrm{U}(1)}) \to \dots ,$$

where we can now identify the spaces $\check{H}^i(M, S_{\mathrm{U}(1)}) \cong H^{i+1}(M, \mathbb{Z})$ using the exponential short exact sequence (2b).

The case of $\mathrm{Spin}^h(n)$-structures becomes more subtle, and we don't have a good generalisation of the previous results, as we have no analogue of the exponential short exact sequences(2b). In particular, the set $\check{H}^1(M, S_{\mathrm{Sp}(1)})$ does not have any natural underlying abelian torsor structure we can exploit. The following result gives a necessary condition for the existence of a $\mathrm{Spin}^h(n)$-structure:

Proposition 2.38 ([AM21]). *Let M^n be an oriented n-dimensional smooth connected manifold. Then M admits a $\mathrm{Spin}^h(n)$-structure if the fourth Stiefel-Whitney class $w_4(M) \in H^4(M, \mathbb{Z}_2)$ is the reduction of an integral class $H^4(M, \mathbb{Z})$.*

It is worth noting that it is known that this obstruction is, in general, not sufficient, but there is little more known in general. We refer the reader to [AM21] for further details in this case.

Let us now give a justification as to why we introduced these groups in a special holonomy course. In summary, we have

Theorem 2.39. *All the groups in the Berger list (cf. Table 1 are subgroups of either $\mathrm{Spin}(n)$, $\mathrm{Spin}^c(n)$ or $\mathrm{Spin}^h(n)$.*

Thus, one has at their disposal all the techniques coming from Spin geometry when one is interested in studying special holonomy metrics. Let us go through each group on the list in detail. To do so, recall the topology result

Proposition 2.40. *Given a covering space $p : E \to B$ and a map $f : X \to B$ with X path-connected and locally path-connected. Then a lift $\tilde{f} : X \to E$ of f exists if and only if*

$$f_*(\pi_1(X)) \subseteq p_*(\pi_1(E)).$$

---

[3] Note that this is a perfectly valid definition for the second Stiefel-Whitney. However, it is not the standard one (cf. [MS74]). One then needs to argue that the two definitions coincide.

First, we have

Proposition 2.41. *The groups* $\mathrm{SU}(n)$*,* $\mathrm{Sp}(n)$*,* $G_2 \subseteq \mathrm{SO}(7)$ *and* $\mathrm{Spin}(7) \subseteq \mathrm{SO}(8)$ *are subgroups of the corresponding* $\mathrm{Spin}(n)$ *group.*

Proof. The aforementioned groups are simply connected for all n. Hence, the embedding map $\iota : G \to \mathrm{SO}(n)$ lifts to the universal cover, $\mathrm{Spin}(n)$, as needed, by Proposition 2.40 $\square$

We are left with the groups $U(n)$ and $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$. We have

Lemma 2.42. *For all* $n \geq 2$*, the unitary groups* $\mathrm{U}(n)$ *are a subgroup of* $\mathrm{Spin}^c(2n)$*.*

Proof. Note that we have a double covering map

$$\begin{aligned} \pi : \mathrm{Spin}^c(n) &\to \mathrm{SO}(n) \times \mathrm{U}(1) \\ [A, \lambda] &\mapsto (\pi_{\mathrm{Spin}}(A), \lambda^2) \end{aligned}$$

with $\pi_{\mathrm{Spin}} : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ the natural projection map. By virtue of Proposition 2.40, the natural map

$$\begin{aligned} \rho : \mathrm{U}(n) &\to \mathrm{SO}(2n) \times \mathrm{U}(1) \\ A &\mapsto (A, \det(A)) \end{aligned}$$

lifts to a map $\hat{\rho} : \mathrm{U}(n) \to \mathrm{Spin}^c(2n)$ provided that the condition on the fundamental groups is satisfied. But this follows from the fact that the determinant map $\det : \mathrm{U}(n) \to \mathrm{U}(1)$ induces an isomorphism on fundamental groups. $\square$

For $n = 1$ the result can be improved slightly, since $\mathrm{U}(1) \cong \mathrm{SO}(2)$, so it embedds in $\mathrm{Spin}(2)$.

Lemma 2.43. *For all* $n \geq 2$*, the groups* $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$ *are a subgroup of* $\mathrm{Spin}^h(4n)$*.*

Proof. We have argued that $\mathrm{Sp}(n) \subseteq \mathrm{Spin}(4n)$, and so $\mathrm{Sp}(n) \times \mathrm{Sp}(1) \subseteq \mathrm{Spin}(4n) \times \mathrm{Sp}(1)$. Taking the $\mathbb{Z}/2\mathbb{Z}$-quotient by the involution $(\mathrm{Id}, \mathrm{Id}) \mapsto (-\mathrm{Id}, -\mathrm{Id})$, we have $Sp(n)Sp(1) \subseteq \mathrm{Spin}^h(4n)$, as needed. $\square$

As in the previous case, we have an exceptional embedding when $n = 1$. We have $\mathrm{Sp}(1)\,\mathrm{Sp}(1) \cong \mathrm{SO}(4)$, so it embedds in $\mathrm{Spin}(4)$.

2.4 Automorphisms of G-strucutres (I)

Let us now briefly discuss the collection of automorphisms of a given G-structure on M. The group of diffeomorphisms $\mathrm{Diff}(M)$ acts naturally on the frame bundle $\mathrm{Fr}(M)$ by pull-back. Given a reduction P of the frame bundle to a principal G-bundle (i.e. a G-structure), we are interested in understanding the stabiliser of P under the diffeomorphism action; the group of symmetries of the G-structures. For instance, if M^n is equipped with a Riemannian metric g, we are computing the isometry group of the metric:

$$\mathrm{Aut}_{\mathrm{SO}(n)}(M) \cong \{\phi \in \mathrm{Diff}(M) \big| \, \phi^* g = g\} \ .$$

It is worth mentioning that this subgroup is a priori a subgroup of the "classical" isometry of (M, g) when thought of as a metric space. The fact that this is, in fact, the whole isometry group is a result of Myers and Steenrod.

We now give some sufficient conditions for the automorphism group to be a Lie group. We will discuss another in Section 4.2. First, it is convenient to introduce

Definition 2.44. A Lie algebra $\mathfrak{g} \subseteq \mathfrak{gl}(n, \mathbb{R})$ is *elliptic* if $\mathfrak{g}$ contains no elements of rank one. A Lie group is called elliptic if its Lie algebra is.

Theorem 2.45 (Ochiai [Och66])**.** *Let G be an algebraic elliptic Lie group and M be a closed manifold equipped with a G-structure, denoted $\mathrm{Fr}_G M$. Then the group $\mathrm{Aut}(\mathrm{Fr}_G M) \subseteq \mathrm{Diff}(M)$ is a Lie group.*

Proof. As the chosen terminology suggests, we want to characterise the ideal of $\mathfrak{X}(M)$ that (infinitesimally) preserves the G-structure as the kernel of an elliptic operator and then use the fact that elliptic operators are Fredholm. In particular, they have a finite-dimensional kernel.

[ADD AN APPENDIX WITH THE MAIN ELLIPTIC THEORY RESULTS].

Note that elliptic regularity ensures that there is no ambiguity arising from our choice of Hölder/Sobolev completion. Since G is algebraic, there exist finitely many tensors $T_1, \ldots, T_k$ that characterise the G-strucutre. The diffeomorphism group acts by pull-back:

$$\begin{aligned} \Phi : \mathrm{Diff}(M) \times \mathcal{Q} &\to \mathcal{Q} \\ (\phi\, ; T_1, \ldots T_k) &\mapsto (\phi^* T_1, \ldots, \phi^* T_k) \ . \end{aligned}$$

The differential of this map at a point $\mathcal{T} \in \mathcal{Q}$ is, by definition,

$$\begin{aligned} \Phi_* : \mathfrak{X}(M) &\to \bigoplus_{i=1}^{k} \Gamma(T^{r_i, s_i} M) \\ X &\mapsto (\mathcal{L}_X T_1, \ldots, \mathcal{L}_X T_k) \ . \end{aligned}$$

We are therefore interested in proving that Φ_* is elliptic, so $\ker \Phi_*$ is finite. We need to show that the symbol of Φ_* is elliptic. Let us compute the symbol of $P = \mathcal{L}_X T$ for a tensor $T \in \mathcal{T}^{r,s}(M)$. Using the algebraic properties of the Lie derivative (cf. Exercise 14), we have that its symbol at a point $p \in M$ and $\xi \in T_p^* M$ is

$$\begin{aligned} \sigma_P(\xi) : T_p M &\to T^{r,s} M \\ X &\mapsto (X \otimes \xi)_* T \ , \end{aligned}$$

where $(X \otimes \xi) \in \mathrm{End}(T_p M)$ is the endomrphism $Y \mapsto \xi(Y) X$, and the action on T is the natural infinitesimal action of $\mathfrak{gl}(n, \mathbb{R})$ on tensors induced by the standard $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$. Thus, $\sigma_P(\xi)$ is injective for all $\xi \neq 0$ if and only if $\mathfrak{stab}(T)$ contains no nonzero rank-one endomorphisms, i.e. it is elliptic.

The argument generalises to multiple tensors T_i directly. Since the G-structure is defined by these tensors, the Lie algebra $\mathfrak{g}$ is precisely the Lie algebra of the Lie group that stabilises the family $\{T_i\}$. Now, Proposition **??** implies Φ_* has a finite-dimensional kernel, and, by exponentiation, we have that $\operatorname{Aut}(\operatorname{Fr}_G M)$ is a Lie group. □

The power of Theorem 2.45 resides in the fact that the ellipticity condition for a Lie algebra is an easily verifiable linear algebra constraint. With respect to the examples of G-strucutres discussed in Section 2.2, we have

Proposition 2.46. *The Lie algebras* $\mathfrak{gl}(n, \mathbb{C})$*,* $\mathfrak{sp}(2n, \mathbb{R})$ *and* $\mathfrak{o}(n)$*, as well as all their subalgebras, are elliptic. The Lie algebra* $\mathfrak{gl}(p, \mathbb{R}) \oplus \mathfrak{gl}(n-p, \mathbb{R}) \subseteq \mathfrak{gl}(n, \mathbb{R})$ *is not elliptic.*

Proof. Exercise 15. □

It is worth noting that Ochiai [Och66] does not require the group G to be algebraic. However, we are mostly interested in algebraic G-structures and the presentation is slightly more straightforward and insightful in this case. The proof presented is virtually the same as that given by Ochiai in his original paper [4]. The condition that the group G is elliptic is, of course, necessary, as seen by considering $G = \mathrm{GL}(n, \mathbb{R})$ or $\mathrm{SL}(n, \mathbb{R})$. The closedness hypothesis for M is also generally necessary. Indeed, the map

$$\begin{aligned} \psi : \mathbb{C}^2 &\to \mathbb{C}^2 \\ (z, w) &\mapsto (z, w + f(z)) \end{aligned}$$

is in $\operatorname{Fr}_{GL(2,\mathbb{C})} \mathbb{C}^2$ for any entire function f, so the automorphism group of the (almost-)complex manifold $\mathbb{C}^2$ is not a Lie group. As we shall see in Section 4.2, one may weaken the compactness requirement if one imposes further constraints on the Lie algebra.

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[4] One of the reasons I prefer this proof as opposed to Ochiai's is that we can very straightforwardly avoid choosing coordinates and work in a coordinate-free setup. I believe one should be able to do the same in Ochiai's framework, but it is not immediately clear to me how to rewrite the details (one needs to replace the tensors T_i by the Jacobian maps of a trivialisation). The reader comfortable with the material is invited to try it. I would very much like to see it.