Lecture Notes on Complex Geometry

Enric Solé-Farré

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Contents

These lecture notes originate from a graduate course I taught in the autumn of 2025 at the Beijing Institute of Mathematical Sciences and Applications (BIMSA) during my tenure as Chern Instructor.

The main goal of the course was to provide a gentle introduction to the vast subject of complex manifolds and their geometry. The main result covered in detail in the notes is the Kodaira embedding theorem, which characterises which manifolds can be embedded as complex submanifolds of a complex projective space. Regrettably, several important results fell beyond the scope of this course due to time constraints. Notable omissions I would have liked to include are the holomorphic Lefschetz fixed point theorem and Chow's theorem. Moreover, the lecture notes focus mostly on proving general results and have few examples (mildly corrected by the exercises).

The first eight sections lean heavily on two outstanding texts: the book by Daniel Huybrechts [Huy05] and the unpublished manuscript by the late Jean-Pierre Demailly [Dem12], available only in online draft form. Additional insights and explanations have been drawn from the works of Andrei Moroianu [Mor07], Raymond Wells [Wel08], and the lecture notes of Dominic Joyce [Joy20].

Sections nine and ten deviate from [Huy05] and take a more differential approach to the topic. While we do not follow a specific reference, they are written in the same spirit as [DK90, Sect. 6] and [Kod86], which the reader is encouraged to consult for a deeper exploration of the topics.

I am convinced that there are a (n embarrassingly large) number of typos and minor mistakes in these notes. I profusely apologise for that. If you find any, please let me know by dropping me an email at esolefarre@bimsa.cn.

Acknowledgements

I would like to thank Dr. Jonas Stelzig and Prof. Dieter Kotschick, who first introduced me to the world of complex geometry and its many wonders.

1 Holomorphic functions: Local theory

We begin by reviewing fundamental properties of holomorphic functions and their generalisation to several complex variables. We identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$ as real vector spaces via the map $(z_1, \ldots, z_n) \mapsto (x_1, y_1, \ldots, x_n, y_n)$ where $z_j = x_j + iy_j$.

Definition 1.1. A function $f : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$ is *differentiable at z_0* if there exists a linear map Df_{z_0} such that

$$f(z) = f(z_0) + Df_{z_0}(z - z_0) + o(\|z - z_0\|).$$

Definition 1.2. A function $f : \mathbb{C}^n \to \mathbb{C}^m$ is *holomorphic at z_0* if it is real-differentiable and its differential Df_{z_0} is complex-linear, i.e., $Df_{z_0} \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^m) \cong \mathrm{GL}(n, \mathbb{C}) \subseteq \mathrm{GL}(2n, \mathbb{R})$.

The complex-linearity condition can be expressed using the standard complex structures J_n and J_m on $\mathbb{R}^{2n}$ and $\mathbb{R}^{2m}$ respectively, that correspond to multiplication by i under a choice of isomorphism $\mathbb{C} \cong \mathbb{R}^2$:

$$J_m \circ Df_{z_0} = Df_{z_0} \circ J_n. \tag{1}$$

This is the coordinate-free form of the *Cauchy-Riemann equations*.

In coordinates $z_j = x_j + iy_j$ and $f = (u_1 + iv_1, \ldots, u_m + iv_m)$, equation (1) becomes:

$$\begin{cases} \partial_{x_j} u_k = \partial_{y_j} v_k \\ \partial_{y_j} u_k = -\partial_{x_j} v_k \end{cases} \qquad \text{for } j = 1, \ldots, n; k = 1, \ldots, m.$$

A powerful reformulation uses the *Wirtinger operators*:

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \overline{z_j}} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

Lemma 1.3. *The Wirtinger operators satisfy:*

(i) $\dfrac{\partial f}{\partial z_j} = \overline{\left(\dfrac{\partial \overline{f}}{\partial \overline{z_j}}\right)}$

(ii) $\dfrac{\partial z_k}{\partial z_j} = \delta_{jk}$, $\dfrac{\partial \overline{z_k}}{\partial z_j} = 0$

(iii) For $f = (f_1, \ldots, f_m)$ and $g = (g_1, \ldots, g_n)$:

$$\frac{\partial (f \circ g)}{\partial z_j} = \sum_{k=1}^{m} \left(\frac{\partial f}{\partial w_k}\frac{\partial g_k}{\partial z_j} + \frac{\partial f}{\partial \overline{w_k}}\frac{\partial \overline{g_k}}{\partial z_j} \right)$$
$$\frac{\partial (f \circ g)}{\partial \overline{z_j}} = \sum_{k=1}^{m} \left(\frac{\partial f}{\partial w_k}\frac{\partial g_k}{\partial \overline{z_j}} + \frac{\partial f}{\partial \overline{w_k}}\overline{\left(\frac{\partial g_k}{\partial z_j}\right)} \right)$$

Moreover, f is holomorphic if and only if $\dfrac{\partial f}{\partial \overline{z_j}} = 0$ for all j.

We can consider the complexified derivative

$$Df(z_0)^{\mathbb{C}} : T_{z_0}\mathbb{R}^{2n} \otimes \mathbb{C} \longrightarrow T_{f(z_0)}\mathbb{R}^2 \otimes \mathbb{C}.$$

The space $T_{z_0}\mathbb{R}^{2n} \otimes \mathbb{C}$ (resp. $T_{f(z_0)}\mathbb{R}^n \otimes \mathbb{C}$) admits the canonical coordinate base $\{\partial/\partial z_i, \partial/\partial \overline{z_i}\}$ (resp. $\{\partial/\partial w, \partial/\partial \overline{w}\}$). In this base, the Jacobian in block form takes the form

The a holomorphic map f, the matrix of derivatives has the form

$$Df = \begin{pmatrix} \frac{\partial f}{\partial z_i} & 0 \\ 0 & \overline{\frac{\partial f}{\partial z_i}} \end{pmatrix} ,$$

reflecting complex-linearity (no $\partial/\partial \bar{z}$-components) of f. It follows that for any holomorphic function f, $\det\left(Df(z_0)^{\mathbb{C}}\right)$ is real and non-negative; $\det\left(Df(z_0)\right) \geq 0$.

Definition 1.4. A holomorphic map $f : U \to V$ is called *biholomorphic* if there exists a holomorphic inverse g to f.

If f is holomorphic and regular (non-degenerate Jacobian), then its Jacobian determinant satisfies

$$\det Df = \left| \det\left(\frac{\partial f}{\partial z_i}\right) \right|^2 > 0.$$

In particular, $\det(Df) \neq 0$ is the local invertibility criterion. Indeed, we have the holomorphic version of the inverse function theorem:

Theorem 1.5 (Holomorphic Inverse Function Theorem)**.** *Let $U, V \subseteq \mathbb{C}^n$ open and $f : U \to V$ a holomorphic map. Consider $z_0 \in U$ such that $\det(Df(z_0) \neq 0$. Then there exist open subsets $z_0 \in U' \subseteq U$ and $f(z_0) \in V' \subseteq V$ such that f restricts to a biholomorphism.*

More generally, a holomorphic map $f : U \to V$ is called a regular (submersion/immersion as appropriate) when the complex-linear partials $\{\partial f/\partial z_i\}_{i=1}^n$ are surjective (or injective) as needed.

Theorem 1.6 (Holomorphic Implicit Function Theorem)**.** *Let $U \subseteq \mathbb{C}^n$ and $V \subseteq \mathbb{C}^m$ be open sets with $n > m$ and $f : U \to V$ a holomorphic function. Assume that there is z_0 such that $Df(z_0)$ satisfies*

$$\det\left[\left(\frac{\partial f_i}{\partial z_j}\right)_{i,j=1,\ldots n}\right] \neq 0\,. \tag{2}$$

Then there exists open sets $U_1 \subseteq \mathbb{C}^{n-m}$, $U_2 \subseteq \mathbb{C}^m$ such that $U_1 \times U_2 \subseteq U$ and a holomorphic function $g : U_1 \to U_2$ satisfying $f(w, g(w)) = f(z_0)$ for all $w \in U_1$.

Proof. The inverse function theorem guarantees the existence and differentiability of g. We need to show that g is holomorphic. Indeed, by the chain rule of Lemma 1.3, we have

$$0 = \frac{\partial}{\partial \overline{w_j}}\Big[f_i(w, g(w))\Big] = \frac{\partial f_i}{\partial \overline{w_j}} + \sum_{k=1}^n \frac{\partial f_i}{\partial z_k}\frac{\partial g_k}{\partial \overline{w_j}} + \frac{\partial f_i}{\partial \overline{z_k}}\overline{\left(\frac{\partial g_k}{\partial w_j}\right)} = \sum_{k=1}^n \frac{\partial f_i}{\partial z_k}\frac{\partial g_k}{\partial \overline{w_j}}\ ,$$

where the first and third terms in the middle line vanish since f is holomorphic.

But the condition in Equation (2) implies that $\left(\frac{\partial f_i}{\partial z_j}\right)$ is invertible, so the only way the second line can vanish is if $\frac{\partial g}{\partial \overline{z_j}} = 0$, as needed. □

A straightforward corollary of the Holomorphic Implicit Function Theorem is the existence of left (resp. right) holomorphic inverses. We have

Corollary 1.7. *Let $U \subseteq \mathbb{C}^n$ and $V \subseteq \mathbb{C}^m$ be open sets and $f : U \to V$ a holomorphic function. Assume we have $z_0 \in U$ such that $Df(z_0)$ has maximal rank. Then,*

(i) If $n > m$, there exists open sets $z_0 \in U' \subset U$ and $V' \subseteq V$, and a biholomorphic map $g : V' \to U'$ such that $f \circ g = \mathrm{Id}$ in V'.

(ii) If $n < m$, there exists open sets $U' \subset U$ and $f(z_0) \in V' \subseteq V$, and a biholomorphic map $g : V' \to U'$ such that $g \circ f = (\mathrm{Id}_n, 0)$ in U'.

1.1 Cauchy Integral Formula and power series expansion

Recall that a key result of complex analysis is the integral formula of Cauchy:

Theorem 1.8 (Cauchy Integral Formula)**.** *Let $K \subseteq \mathbb{C}$ be a compact subset with piecewise C^1 boundary $C = \partial K$, and $f : K \to \mathbb{C}$ a differentiable function. Then for $z \in K \setminus \partial K$, we have*

$$2\pi i f(z) = \int_{\partial K} \frac{f(w,\overline{w})}{w-z} dw + \int_K \frac{\partial f}{\partial \overline{w}} \frac{dw \wedge d\overline{w}}{w-z} \tag{3}$$

Proof. Without loss of generality, we assume $z = 0$. We want to study the function $f(w,\overline{w})/w \in L^1(K)$. Taking $\delta > 0$, we have on one side,

$$\int_{K \setminus B_\delta(0)} d\left(\frac{f(w,\overline{w})}{w}\right) dz = -\int_{K \setminus B_\delta(0)} \frac{\partial f}{\partial \overline{w}} \frac{dw \wedge d\overline{w}}{w} \ .$$

On the other side, by Stokes' theorem, we get

$$\int_{K \setminus B_\delta(0)} d\left(\frac{f(w,\overline{w})}{w}\right) dw = \int_{\partial K} \frac{f(w,\overline{w})}{w} dw - \int_{\partial B_\delta} \frac{f(w,\overline{w})}{w} dw \ .$$

Parametrising the last term in polar coordinates $w = \delta e^{i\theta}$, we have

$$\int_{\partial B_\delta} \frac{f(w,\overline{w})}{w} = \int_0^{2\pi} f(\delta,\theta) i d\theta \ .$$

Putting everything together and taking δ to zero, the claim follows by continuity of f. □

Of course, we are mostly interested in the case where f is holomorphic, so the last term in (3) vanishes, and we have the usual expression

$$f(z) = \frac{1}{2\pi i} \int_{\partial K} \frac{f(w)}{w-z} dw \tag{4}$$

The Cauchy Integral Formula (CIF) generalises to higher dimensions by considering polydiscs $D_R(w) = B_{R_1}(w_1) \times \dots B_{R_n}(w_n)$ and iterative use of Fubini's theorem:

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial D_R(z)} \frac{f(w_1,\dots,w_n)}{(w_1-z_1)\dots(w_n-z_n)} dw_1 \dots dw_n \ .$$

The CIF has some important, remarkable consequences for the regularity of the function f:

Proposition 1.9. *Let $f : U \to \mathbb{C}$ be a holomorphic function. Then f is analytic. That is, it admits a convergent power series expansion*

$$2\pi i f(z) = \sum_{|\alpha|\geq 0} \frac{f^{(\alpha)}(z_0)}{\alpha!} z^\alpha \ ,$$

with α a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, .

Proof. We argue the case $n = 1$; the higher-dimensional case follows. We know $\dfrac{1}{z-w} = \dfrac{1}{z}\dfrac{1}{(1-w/z)} = \displaystyle\sum_{k\geq 0} \frac{w^k}{z^{k+1}}$ for $|w| < |z|$. Substituting in the CIF and using Lebesgue monotone convergence, we have

$$2\pi i f(w) = \int_C \frac{f(z)}{z-w} dz = \int_C \sum_{k\geq 0} w^k \frac{f(z)}{z^{k+1}} dz = \sum_{k\geq 0} w^k \int_C \frac{f(z)}{z^{k+1}} dz \ .$$

Analyticity follows. The coefficients of the power expansion are the successive derivatives of f by the uniqueness of Taylor expansions. Alternatively, one can check directly:

$$\begin{aligned} f'(w) &= \lim_{h\to 0} \frac{f(w+h) - f(w)}{h} = \lim_{h\to 0} \frac{1}{2\pi i h} \int_C \frac{f(z)}{z-(w+h)} - \frac{f(z)}{z-w} dz \\ &= \lim_{h\to 0} \frac{1}{2\pi i h} \int_C \frac{h f(z)}{(z-w-h)(z-w)} dz = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-w)^2} dz \ . \end{aligned}$$

□

The analyticity of holomorphic functions has some remarkable consequences:

Theorem 1.10 (Open mapping theorem)**.** *Let $f : U \to \mathbb{C}$ be a non-constant holomorphic function on an open set U. The f is an open mapping.*

In particular, if there exists $z_0 \in U$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in U$, f is constant.

Theorem 1.11 (Identity principle)**.** *Let U be an open connected subset of $\mathbb{C}^n$ and $f, g : U \to \mathbb{C}$ holomorphic functions. If $f = g$ on an open subset $V \subset U$, then $f \equiv g$ on all of U.*

Proof. Let

$$W = \left\{ z \in U \;\middle|\; \frac{\partial^\alpha f}{\partial z^\alpha} = \frac{\partial^\alpha g}{\partial z^\alpha} \quad \forall\, \alpha \text{ multi-index} \right\} .$$

The set W is clearly closed and non-empty. By analyticity, W is also open, and by connectedness, $W = U$. □

Another consequence of the Cauchy Integral Formula, Equation (4), is

Lemma 1.12 (Cauchy inequality)**.** *Let $f : U \to \mathbb{C}$ be a holomorphic function and take $R > 0$ such that the ball $B_R(z_0)$ is contained in U. Then*

$$|f^{(\alpha)}(z_0)| \leq \frac{\alpha!}{R^\alpha} \sup_{\partial B_R(z_0)} |f(z)| \tag{5}$$

There are two important corollaries of this inequality:

Theorem 1.13 (Generalised Liouville theorem)**.** *Let $f : \mathbb{C}^n \to \mathbb{C}$ a holomorphic function such that $|f(z)| \leq C(1 + |z|)^D$ for some $C, D \geq 0$. Then f is a polynomial with degree at most D.*

Theorem 1.14 (Montel's theorem)**.** *Let $U \subseteq \mathbb{C}^n$ open, and consider $\mathcal{O}(U)$ the space of holomorphic functions on U, equipped with the uniform convergence on compact sets topology, induced by $C^0(U)$. Then every locally uniformly bounded sequence $(f_j)_j \subseteq \mathcal{O}(U)$ has a convergent subsequence.*

Proof. By Arzelà–Ascoli. □

1.2 Hartogs' phenomenon and the Weierstrass theorems

So far, all properties that we have discussed are direct analogues of properties that occur in complex analysis ($n = 1$) and have discussed the rigidity of holomorphic functions. First, we need the following technical lemma

Lemma 1.15. *Consider the open cylinder $U \times V$ with $U \subseteq \mathbb{C}^n$ open, and $V \subseteq \mathbb{C}$ a neighbourhood of $\partial B_\varepsilon(z_0)$ and let $f : V \times U \to \mathbb{C}$ a holomorphic function. Then*

$$g(z_1, \dots, z_n) := \int_{\partial B_\varepsilon(z_0)} f(\xi, z_1, \dots z_n) d\xi$$

is a holomorphic function on U.

Proof. Notice that if f were holomorphic on $U \times B_\varepsilon(z_0)$, we would essentially be done. The idea is to reduce it to an equivalent situation.

Since $\partial B_\varepsilon(z_0)$ is compact, for every $\delta > 0$, there exists finitely many ξ_i such that $\{B_\delta(\xi_i)\}$ cover $\partial B_\varepsilon(z_0)$. By choosing δ small enough, we can ensure $B_\delta(\xi_i) \subseteq V$ and f has a convergent power series in $B_\delta(\xi_i) \times U_i$ for all i.

We can now split the integral into a finite sum of integrals where f has a power series expansion. □

Let us now focus on the extension problem.

Theorem 1.16 (Hartogs' principle)**.** *Let $D_R(0)$ and $D_{R'}(0)$ be two polydiscs in $\mathbb{C}^n$ with $\overline{D_{R'}(0)} \subseteq D_R(0)$ so $R_i > R'_i$ for all i. Any holomorphic function $f : D_R(0) \setminus \overline{D_{R'}(0)} \to \mathbb{C}$ can be uniquely extended to a holomorphic function $\overline{f} : D_R(0) \to \mathbb{C}$.*

Proof. Let $w = (z_2, \dots, z_n)$ with $|z_2| > R'_2$. We can use the Cauchy formula for the function $z \mapsto f(z, w)$, for $R'_1 < \delta < R_1$:

$$f(z, w) = \frac{1}{2\pi i} \int_{|\xi|=\delta} \frac{f(\xi, w)}{(\xi - z)} d\xi$$

The integrand is $(\xi, z, w) \mapsto \frac{f(\xi,w)}{(\xi-z)}\, d\xi$, which is holomorphic on $B_c(\delta) \times B_{\delta-c}(0) \times D_{R_2,\dots,\ R_n}(0)$ for some small c. Therefore, by the lemma, the function

$$\widetilde{f}(z,w) = \frac{1}{2\pi i} \int_{|\xi|=\delta} \frac{f(\xi,w)}{(\xi-z)}\, d\xi$$

is holomorphic on $B_{\delta-c}(0) \times D_{R_2,\dots,R_n}$, providing the desired extension by the identity principle. □

We conclude this subsection by proving two technical lemmas, due to Weierstrass, that will be useful throughout the course. First, we need

Definition 1.17 (Weierstrass Polynomial)**.** A *Weierstrass polynomial* in z_1 of degree d is a polynomial

$$z_1^d + a_1(w) z_1^{d-1} + \cdots + a_d(w),$$

where $a_i(z')$ are holomorphic functions in $w = (z_2, \dots, z_n)$ defined in a neighbourhood of the origin and such that $a_i(0, \dots, 0) = 0$.

Theorem 1.18 (Weierstrass Preparation Theorem)**.** *Let $f : D_\varepsilon(0) \to \mathbb{C}$ with $f(0,0) = 0$ and $f(z_1, 0, \dots, 0) \not\equiv 0$. Then for some smaller ball $D_{\varepsilon'}(0)$ there exists a unique decomposition:*

$$f = g \cdot h$$

where g is a Weierstrass polynomial in z_1, and $h : D_{\varepsilon'}(0) \to \mathbb{C}$ is a holomorphic function without zeros.

Proof. By taking ε_1 smaller if needed, we may assume $f(z_1, 0, \dots, 0)$ vanishes only at 0, with multiplicity d. Moreover, choose $r \in (0, \varepsilon)$ and $\varepsilon_2, \dots, \varepsilon_n$ so that $f(z_1, w) \neq 0$ for $|z_1 - r| < \varepsilon$ and $|w_i| < \varepsilon_i$, which exist by continuity and compactness.

For small w, the zeros of $f_w(z) = f(z, w)$ are given by $a_1(w), \dots, a_d(w)$. Define:

$$g(z,w) = \prod_{i=1}^{d} (z_1 - a_i(w)), \quad h = \frac{f}{g}$$

We need to show that g and h are holomorphic in z_1 and w. Holomorphicity in z_1 is straightforward.

To see g is holomorphic in w, notice that this amounts to showing that the elementary symmetric polynomials in terms of $a_i(w)$ are holomorphic, which are linear combinations of $S_k = \sum_{i=1}^n a_i(w)^k$ for $k = 0, \dots, d$. By the Cauchy residue formula [1], we have

$$\sum_{i=1}^{n} a_i(w)^k = \frac{1}{2\pi i} \int_{|\xi|=\varepsilon_1} \xi^k \frac{\partial}{\partial \xi} \Big[\log \big(f(\xi, w) \big) \Big] d\xi \ ,$$

which is holomorphic by Lemma 1.15. Finally, we may write

$$h(z_1, w) = \frac{1}{2\pi i} \int_{|\xi|=\varepsilon_1'} \frac{h(\xi, w)}{\xi - z_1} d\xi,$$

which is everywhere holomorphic by Lemma 1.15 and f/g being holomorphic on the annulus. □

[1] Check this formula by yourself, note that $k = 0$ is precisely the argument principle, giving the count of zeros enclosed in the domain.

Theorem 1.19 (Weierstrass Division Theorem). *Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$, and let g be a Weierstrass polynomial of degree d. Then there exist a unique $h \in \mathcal{O}_{\mathbb{C}^n,0}$ and $r \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ with $\deg r < d$ such that:*

$$f = g \cdot h + r$$

Proof. Define

$$h(z, w) = \frac{1}{2\pi i} \int_{\partial B_\varepsilon(0)} \frac{f(\xi, w)}{g(\xi, w)} \frac{d\xi}{\xi - z}$$

and check that $r = f - gh$ lies in $\mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ and is of degree $< d$ holomorphicity. □

1.3 The ring of holomorphic germs $\mathcal{O}_{\mathbb{C}^n,0}$ and Hilbert's Nullstellensatz

We study the local behaviour of holomorphic functions on an arbitrarily small neighbourhood of a point. More formally, this leads to considering the notion of germs and stalks:

Definition 1.20. The *holomorphic stalk at the origin*, denoted $\mathcal{O}_{\mathbb{C}^n,0}$, is the set of all equivalence classes of pairs (U, f), where U is an open neighbourhood of 0 in $\mathbb{C}^n$ and $f : U \to \mathbb{C}$ is a holomorphic function.

Two pairs (U, f) and (V, g) are considered equivalent if there exists an open neighbourhood $W \subseteq U \cap V$ of 0 such that f and g agree on W:

$$(U, f) \sim (V, g) \iff f|_W = g|_W \text{ for some open } W \ni 0.$$

An equivalence class is called a *holomorphic germ at* 0.

Alternatively, one can think of the holomorphic stalk as the set of convergent power series inside $\mathbb{C}[[z_1, ..., z_n]]$, (cf. Exercise 9).

Remark 1.21. Definition 1.20 might feel overly complicated and slightly unnatural. Indeed, stalks and germs are better understood in the language of sheaves, which we will introduce in Section 3.

The holomorphic stalk $\mathcal{O}_{\mathbb{C}^n,0}$ inherits a ring structure from that of holomorphic functions. We devote ourselves to studying its structure. We prove

Theorem 1.22. *The stalk of holomorphic germs $\mathcal{O}_{\mathbb{C}^n,0}$ is*

(i) a local ring,

(ii) a unique factorisation domain (UFD), and

(iii) Noetherian.

Proof. (i) The ideal $\mathcal{I}_0$ given by (germs of) functions vanishing at the origin is maximal, with residue field $\mathcal{O}_{\mathbb{C}^n,0}/\mathcal{I}_0 \cong \mathbb{C}$. If $f \in \mathcal{O}_{\mathbb{C}^n,0}$ satisfies $f \neq 0$, then one can show with little work that $f \in \mathcal{O}^*_{\mathbb{C}^n,0}$, so there's no other maximal ideal $\mathcal{I}_0$.

(ii) We prove this by induction. The case $n = 0$ is trivial.

Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$ vanishing at the origin. By the Weierstrass Preparation Theorem 1.18, we can uniquely write f as $f = u \cdot p$, with $u \in \mathcal{O}^{\times}_{\mathbb{C}^n,0}$ a unit and $p \in \mathcal{O}_{\mathbb{C}^{n-1},0}[w]$ (the germ of) a Weierstrass polynomial.

The $\mathcal{O}_{\mathbb{C}^{n-1},0}$ is a UFD by induction hypothesis, and so is $\mathcal{O}_{\mathbb{C}^{n-1},0}[w]$ by Gauss' lemma.

It remains to check that p is a finite irreducible element of $\mathcal{O}_{\mathbb{C}^n,0}$, which is straightforward using the uniqueness of the decomposition of the Weierstrass Preparation Theorem 1.18.

(iii) Again, we prove this by induction, with the case $n = 0$ being immediate.

Assume $\mathcal{O}_{\mathbb{C}^{n-1},0}$ is Noetherian, and therefore so is the subring $\mathcal{O}_{\mathbb{C}^{n-1},0}[z_1] \subseteq \mathcal{O}_{\mathbb{C}^n,0}$, by Hilbert's basis theorem.

Let $I \in \mathcal{O}_{\mathbb{C}^n,0}$ an ideal, so $I \cap \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ is finitely generated.

Take $f \in I$.By the Weierstrass Preparation Theorem 1.18, we get $f = gh$ with $h \in \mathcal{O}^*_{\mathbb{C}^n,0}$ and $g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$, so $g = fh^{-1} \in I \cap \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$.

For any other $\widetilde{f} \in I$, the Weierstrass division theorem implies that $\widetilde{f} = g\widetilde{h} + \widetilde{r}$ for $r \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$. Since $\widetilde{f}$ and g are in I, it follows that $r \in I \cap \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$. Thus, I is a finitely generated ideal.

□

We include one final lemma for the sheaf of holomorphic stalks that will be useful in the future:

Lemma 1.23. *Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$ irreducible. Then for $\varepsilon > 0$ small enough $f \in \mathcal{O}_{\mathbb{C}^n,z}$ is irreducible for all $z \in B_\varepsilon(0)$. Similarly, if $f, g \in \mathcal{O}_{\mathbb{C}^n,0}$ are coprime, they remain coprime in $\mathcal{O}_{\mathbb{C}^n,z}$ for all $z \in B_\varepsilon(0)$ for ε small enough.*

Proof. We include the details for the proof of when f and g are coprime; the proof of irreducibility follows the same logic.

By the Weierstrass Preparation Theorem 1.18, we may assume f and g are Weierstrass polynomials. Thus, they must be coprime as polynomials. By Gauss' lemma, this means we can find polynomials $p_1, p_2 \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ and $0 \neq h \in \mathcal{O}_{\mathbb{C}^{n-1},0}$ such that $h = fp_1 + gp_2$. The claim follows. □

Let us now define analytic sets and their germs. Given $f : U \to \mathbb{C}$ a holomorphic function, we denote its vanishing set as $Z(f) = \{z \in U \big| \, f(z) = 0\}$.

Definition 1.24. An *analytic set* $Z \subseteq X$ is a set such that for each $x \in Z$, there exists an open neighbourhood $U \ni x$ and holomorphic functions $f_1, \ldots, f_k \in \mathcal{O}(U)$ with

$$Z \cap U = Z(f_1, \ldots, f_k) = \bigcap_{i=1}^{k} Z(f_i) \,.$$

In the same spirit as before, we define the corresponding germs

Definition 1.25. An *analytic germ* at $x \in X$ is an equivalence class of analytic sets under the relation $Z_1 \sim Z_2$ if $Z_1 \cap U = Z_2 \cap U$ for some neighbourhood $U \ni x$.

Given a germ X at the origin, we denote by $I(X)$ the set of homomorphic germs s satisfying the condition $X \subseteq Z(s)$. So $Z(\cdot)$ takes holomorphic germs (or functions) to analytic germs, and $I(\cdot)$ takes analytic germs to their holomorphic counterparts. They satisfy the following relations:

Lemma 1.26.

(i) *For any subset $A \subseteq \mathcal{O}_{X,x}$, $Z(A)$ is a well-defined analytic germ with $Z(A) = Z((A)_{\mathcal{O}_{X,x}})$.*

(ii) *For every analytic germ Z, $I(Z) = \{f \in \mathcal{O}_{X,x} \mid Z \subset Z(f)\}$ is an ideal.*

(iii) *If $X_1 \subset X_2$ are analytic germ, then $I(X_2) \subset I(X_1)$. If $I_1 \subset I_2$ are ideals in $\mathcal{O}_{X,x}$, then $Z(I_2) \subset Z(I_1)$.*

(iv) *$Z = Z(I(Z))$ and $I \subset I(Z(I))$.*

(v) *$Z(I \cdot J) = Z(I) \cup Z(J)$ and $Z(I + J) = Z(I) \cap Z(J)$.*

Proof. Exercise. □

The relation between holomorphic and analytic germs is made precise by Hilbert's Nullstellensatz:

Theorem 1.27 (Hilbert's Nullstellensatz Theorem)**.** *For any ideal $I \subseteq \mathcal{O}_{X,x}$, we have:*

$$\sqrt{I} = I\big(Z(I)\big)$$

where $\sqrt{I}$ is the radical ideal of I; $\sqrt{I} = \{f \in \mathcal{O}_{X,x} \mid f^n \in I \text{ for some } n\}$.

We would like to understand the fundamental "building blocks" of holomorphic and analytic germs. Since the holomorphic stalk naturally carries a ring structure, our focus will be on its prime ideals. On the side of analytic germs, we introduce the following definition:

Definition 1.28. An analytic germ is Z called *irreducible* if for any union $Z = Z_1 \cup Z_2$ with Z_i analytic germs, either $Z = Z_1$ or $Z = Z_2$.

As expected, we have the following result

Lemma 1.29. *An analytic germ Z is irreducible if and only if $I(Z)$ is a prime ideal.*

Proof. Let $f_1 f_2 \in I$. Then $Z = \big(Z \cap Z(f_1)\big) \cup \big(Z \cap Z(f_2)\big)$. If Z is irreducible $Z = Z \cap Z(f_i)$, so f_i vanishes along Z, i.e. $f_i \in I(Z)$.

The converse follows similarly. □

2 Complex and almost complex manifolds

We now introduce the main class of objects that we are interested in, complex manifolds. We will give two definitions for them. First, using complex charts and holomorphic transition functions. Second, we adopt a more differential geometric style, using $\mathrm{GL}(n, \mathbb{C})$-structures, more commonly known as almost complex structures on a real manifold. The two definitions are equivalent by virtue of the celebrated Newlander-Nirenberg Theorem.

For the remainder of the notes, a (topological) manifold is a locally Euclidean, second-countable [2], Hausdorff space. Recall from differential geometry:

Definition 2.1. A $\mathcal{C}^k$*-manifold* is a topological manifold equipped with an atlas of charts $(U_i, \phi_i)_{i\in I}$, where transition functions $\phi_{ij} = \phi_i \circ \phi_j^{-1}$ are $\mathcal{C}^k$-diffeomorphisms between open sets in $\mathbb{R}^n$.

Recall that $\mathcal{C}^0$-manifolds are topological manifolds, and that a theorem of Whitney tells us that a $\mathcal{C}^k$-manifold for $k \geq 1$ admits a compatible $\mathcal{C}^\infty$-structure.

There is an intermediate notion between $\mathcal{C}^0$ and $\mathcal{C}^1$, called PL:

Understanding when a manifold admits a smooth structure, and if so, how many, was an active research area in the second half of the 20th century that is nowadays well understood (see e.g. Kervaire–Milnor groups, Kirby–Siebenmann invariants, geometrisation conjecture) except in dimension 4, where surprising links to other areas of mathematics appear.

Another class is the class of affine manifolds, where the $\mathcal{C}^k$ condition is replaced by $\mathrm{Aff}(\mathbb{R}^n)$, requiring the transition maps to be affine maps of $\mathbb{R}^n$. Affine manifolds are quite mysterious, and longstanding conjectures and open problems remain to be tackled.

Definition 2.2. A *complex manifold* is a manifold equipped with an atlas of charts $(U_i, \phi_i)_{i\in I}$, where transition functions $\phi_{ij} = \phi_i \circ \phi_j^{-1}$ are biholomorphisms between open sets in $\mathbb{C}^n$.

To avoid issues and pathologies, we will always assume our atlases are maximal, i.e. they are not a proper subset of any other atlas. Every atlas $\{(U_i, \phi_i) : i \in I\}$ is contained in a unique maximal atlas: the set of all charts (U, ϕ) compatible with (U_i, ϕ_i) for all $i \in I$, so there is no prejudice in always taking the maximal atlas.

We will mostly refer to X as the complex manifold, omitting the atlas to lighten notation, as is typically done in differential geometry. As in the previous case, we can ask the questions:

Question 2.3. *When does a manifold M admit the structure of a complex manifold? Is the complex structure unique? Can we classify complex manifolds up to biholomorphism?*

In contrast to the smooth case, very little is known in this case, beyond some obvious topological constraints, discussed in the exercises.

In the compact setting, some existence and classification results exist for complex dimensions 1 and 2. Already in dimension 3, we find one of the most (in)famous open problems in differential

[2] Sometimes

geometry:

Question 2.4. *Does the round 6-sphere S^6 admit the structure of a complex structure?*

In the non-compact case, we have Liouville-type obstructions, so we know that the complex plane $\mathbb{C}^n$ is not biholomorphic to certain bounded domains (e.g. the unit ball or polydisc). However, there is no high-dimensional analogue of the Uniformisation Theorem. In general, complex domains carry intrinsic complex-analytic invariants that obstruct biholomorphism. For $n > 1$, many bounded domains are not biholomorphically equivalent.

Definition 2.5. Let X be a complex manifold, and $f : X \to \mathbb{C}$ a function. We call f *holomorphic* if, for all charts (U, ϕ) in the (maximal) atlas, $f \circ \phi$ is holomorphic in the sense of Section 1.

Definition 2.6. Let X, Y be complex manifolds and $f : X \to Y$ a continuous function. The map f is said to be holomorphic if for all charts (U, ϕ) of X and (V, ψ) of Y, the map

$$\psi^{-1} \circ f \circ \phi$$

is a holomorphic map in the sense of Section 1.

Definition 2.7. Let X be a complex manifold of dimension n, and $Y \subseteq X$.

We say Y is an (embedded) *complex submanifold* of X of dimension k if for each $y \in Y$ there exist an open neighbourhood U of y and local holomorphic coordinates $(z_1, \ldots, z_n)$ on U such that $Y = Z(z_{k+1}, \ldots, z_n)$.

We will usually require Y to be closed in X. With the definition above, it is easy to see that

Proposition 2.8. *A complex submanifold is a complex manifold such that the inclusion map $\iota_Y : Y \hookrightarrow X$ is injective and holomorphic.*

Conversely, a holomorphic map $f : Y \to X$ is called an embedding if it is injective, locally closed, and with injective differential $Df : T_yY \to T_{f(y)}X$ for all $y \in Y$. It follows easily that f is an embedding if and only if $f(Y)$ is a complex submanifold of X, biholomorphic to Y.

As in the smooth case, we can produce examples of complex submanifolds via the holomorphic implicit function theorem:

Theorem 2.9. *Let $f : X \to Y$ be a holomorphic map between complex manifolds of dimensions n and m respectively, and let $y \in Y$ such that the differential $Df_x : T_xX \to T_yY$ is surjective for all $x \in f^{-1}(y)$. Then $f^{-1}(y)$ is a complex submanifold of dimension $n - m$.*

A point y satisfying the conditions of the theorem above is called a *regular* point (or value, if $Y = \mathbb{C}$). We have

Corollary 2.10. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function and c a regular value, then $Z(f - c) = f^{-1}(c)$ is a complex hypersurface (complex submanifold) of complex codimension* 1.

Unfortunately, one needs to work a bit harder if one is interested in finding examples of compact complex submanifolds.

Exercise 1. *The only compact complex submanifolds of* $\mathbb{C}^n$ *(when considered as submanifolds of* $\mathbb{C}^n$*) are discrete points.*

Let us introduce the first compact example, which will play a prominent role throughout the course. The complex projective space $\mathbb{CP}^n$ is the moduli space of complex lines (or dually hyperplanes) in $\mathbb{C}^{n+1}$. It can be realised as the quotient

$$\mathbb{CP}^n \cong (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^* ,$$

where the $\mathbb{C}^*$-action is given by $z \mapsto \lambda z$.

The complex projective space $\mathbb{CP}^n$ is a compact n-dimensional complex manifold.

Let us define homogeneous coordinates $[z_0, \dots, z_n]$ on $\mathbb{CP}^n$. For $i = 0, \dots, n$, define a chart (U_i, ϕ_i) on $\mathbb{CP}^n$ by $U_i = \mathbb{C}^n$ and $\phi_i : \mathbb{C}^n \to \mathbb{CP}^n$ given by

$$\phi_i : (w_1, \dots, w_n) \longmapsto [w_1, \dots, w_i, 1, w_{i+1}, \dots, w_n].$$

This is a homeomorphism with the open subset

$$\phi_i(U_i) = \{[z_0, \dots, z_n] \in \mathbb{CP}^n : z_i \neq 0\} \text{ in } \mathbb{CP}^n.$$

For $0 \leq i < j \leq n$, the transition function $\phi_{ij} = \phi_j^{-1} \circ \phi_i$ is given by

$$\begin{aligned} \phi_{ij} : \mathbb{C}^n \setminus \{z_j = 0\} &\to \mathbb{C}^n \setminus \{z_i = 0\} \\ (z_1, \dots, z_n) &\longmapsto \left(\frac{z_1}{z_j}, \dots, \frac{z_i}{z_j}, \frac{1}{z_j}, \frac{z_{i+1}}{z_j}, \dots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \dots, \frac{z_n}{z_j}\right) . \end{aligned}$$

The ϕ_{ij}'s are clearly biholomorphisms. So $\{(U_i, \phi_i)\}_{i=0,\dots,n+1}$ forms an atlas of $\mathbb{CP}^n$, that extends to the corresponding maximal atlas.

Now, we have the following example of complex submanifolds:

Proposition 2.11. *Let* $p : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}$ *a homogeneous polynomial such that* 0 *is a regular value of* p*, and consider*

$$X = \{[z_0, \dots, z_n] \in \mathbb{CP}^n \big| \, (z_0, \dots, z_n) \in Z(p)\} .$$

Then X *is a well-defined compact complex submanifold of* $\mathbb{CP}^n$*.*

Proof. X is well-defined, since p is homogeneous, so $p(z) = 0$ implies $p(\lambda z) = 0$ for all $\lambda \in \mathbb{C}^*$.

Now, X is covered by the charts $V_i = (X \cap U_i)$, where U_i are the standard charts for $\mathbb{CP}^n$ used above. On each V_i, X is described by the vanishing of $p(z_0, \dots, z_{i-1}, 1, z_{i+1}, \dots z_n)$, and Theorem 2.9 concludes the proof. ☐

We give two examples:

Example 2.12. *For* $d \in \mathbb{N}_+$*, the set* $X = (z_0^d + z_1^d + z_2^d) \subseteq \mathbb{CP}^2$ *is a Riemann surface of genus* $g = \frac{(d-1)(d-2)}{2}$*.*

Example 2.13. *The set* $Y = Z(z_0^2 + \cdots + z_3^2) \subseteq \mathbb{CP}^3$ *is a projective complex manifold biholomorphic,* $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Of course, one may ask how general the condition for 0 to be a regular value of a homogeneous polynomial. We leave it as an exercise to show that

Exercise 2. *The set of homogeneous polynomials for which* 0 *is a regular value is generic.*

More generally, one has

Proposition 2.14. *Let* $(p_1, \dots p_k) : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}^k$ *a collection of homogeneous polynomials such that* $(0, \dots, 0)$ *is a regular value. Then* $\big(Z(p_1) \cap \cdots \cap Z(p_n)\big)/\mathbb{C}^* \subseteq \mathbb{CP}^n$ *is a complex submanifold of dimension* $n - k$*, called a* complete intersection*.*

More generally, a projective variety is a subset X of $\mathbb{CP}^n$ which is locally defined by the vanishing of finitely many homogeneous polynomials.

Projective complex manifolds allow us to consider a large number of examples of complex manifolds. Moreover, since they are defined using polynomials, they can be studied using algebraic techniques, giving rise to complex algebraic geometry.

In the opposite direction, one may consider under what conditions one can guarantee that a compact complex manifold X can be realised as a submanifold of a complex projective space. Complex manifolds that can be realised as submanifolds of $\mathbb{CP}^n$ are called projective.

We can fully characterise all projective manifolds, by the Kodaira Embedding Theorem, which we will prove during this course.

Complex Lie groups also provide important examples of complex manifolds:

Definition 2.15. A *complex Lie group* is a group G that is also a complex manifold such that multiplication and inversion are holomorphic maps.

Examples include the general linear groups $\mathrm{GL}_n(\mathbb{C})$, special linear groups $\mathrm{SL}_n(\mathbb{C})$, complex tori, etc.

Proposition 2.16. *Let* G *be a complex Lie group acting holomorphically on a complex manifold* X*. If the action is free and proper, then the quotient* X/G *carries a canonical complex manifold structure for which the projection* $X \to X/G$ *is a holomorphic submersion.*

Proof. See [Wel08, Prop. 5.3]. □

As a direct application of this proposition, we give two further examples of complex manifolds: Hopf and Iwasawa manifolds.

Hopf manifolds are examples of compact complex manifolds obtained as quotients of $\mathbb{C}^n \setminus \{0\}$ by a discrete group generated by contractions. For a concrete example, let $\alpha \in (0, 1)$ and

$$H_A = (\mathbb{C}^n \setminus \{0\}) / \sim_\alpha$$

where $z \sim_\alpha w$ if $z = \alpha^n w$ for some n.

Remark 2.17. Hopf manifolds are diffeomorphic to $S^{2n-1} \times S^1$ (think in polar coordinates) and provide important examples in complex geometry, as we shall see.

Finally consider $\mathbb{U} \subseteq \mathrm{GL}(3, \mathbb{C})$ the subgroup of upper-triangular matrices

$$U = \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix}$$

and its subgroup $\mathbb{U}_{\mathbb{Z}} = \mathbb{U} \cap \mathrm{GL}(3, \mathbb{Z}[i])$. The group $\mathbb{U}_{\mathbb{Z}}$ acts by translations $(w_1, w_2, w_3) \cdot (z_1, z_2, z_3) \mapsto (z_1 + w_1, z_2 + w_2, z_3 + w_3)$, which is a free and proper action, so the quotient is a complex manifold, known as the Iwasawa manifold $\mathbb{I} = \mathbb{U}/\mathbb{U}_{\mathbb{Z}}$.

The first and third coordinate provide a holomorphic submersion $f : \mathbb{I} \to \mathbb{C}/\mathbb{Z}[i] \times \mathbb{C}/\mathbb{Z}[i]$, with the fibres given by the remaining coordinate, biholomorphic to $\mathbb{C}/\mathbb{Z}[i]$.

2.1 Almost complex structures

We now introduce the second definition of complex manifolds, via almost complex structures. The idea is to consider a weaker notion of complex structures and study the relation between the two.

The idea is the following: Let X be a complex n-manifold in the sense of Definition 2.2. Then, the underlying topological manifold carries a natural smooth real $2n$-manifold $X_{\mathbb{R}}$. Its tangent bundle $TX_{\mathbb{R}}$ inherits the structure of a complex vector bundle, which is reflected in the existence of a bundle endomorphism $J \in \mathcal{C}^\infty(\mathrm{End}(TX_{\mathbb{R}})$ such that $J^2 = -\mathrm{Id}_{2n}$ fiberwise. This motivates the notion of an almost complex structure:

Definition 2.18. Let X be a real $2n$-manifold. An *almost complex structure* J on X is the choice of a section J in $\mathcal{C}^\infty(\mathrm{End}(TX_{\mathbb{R}}))$ satisfying the condition $J^2 = -\mathrm{Id}_{2n}$.

A manifold X equipped with an almost complex structure J is called an *almost complex manifold.*

Any complex manifold in the sense of Definition 2.2 induces a real manifold X with an almost complex structure J. The converse is not true, as we shall see.

Since an almost complex structure J furnishes the tangent space with the structure of a complex vector space pointwise, we can define the analogue notions of holomorphic functions and maps.

Definition 2.19. Let (X, J) be an almost complex manifold and $f : X \to \mathbb{C}$ a smooth function. We say f is *J-holomorphic* function if

$$df \circ J = i df \ .$$

Similarly, we have

Definition 2.20. Let (X, I) and (Y, J) be almost complex manifolds and $f : X \to Y$ a smooth map. We say f is a *pseudo-holomorphic* map if

$$df \circ I = J \circ df \ .$$

Before proceeding, let us say a few words about the existence of almost complex structures.

Unlike the case of complex strucutres, we are not requiring that our structure solves any PDEs (the transition maps being holomorphic), just the existence of a special section of the endomorphism bundle $\mathrm{End}(TM)$ (or the reduction of the frame bundle to a principal $\mathrm{GL}(n,\mathbb{C})$-bundle). This problem is well-understood from the point of view of classifying spaces, and it allows us to phrase necessary and sufficient conditions for the existence of an almost complex structure in terms of very explicit topological conditions in low dimensions:

Proposition 2.21. *Let M^{2n} be a closed manifold*

(i) For $n = 1$, M admits an almost complex structure if and only if M is orientable (equiv. $w_1(M) = 0$).

(ii) For $n = 2$, M admits an almost complex structure if and only if M is orientable and there exists $h \in H^2(M,\mathbb{Z})$ such that

$$h^2 = 3\sigma(X) + 2\chi(X) \qquad\qquad h \equiv_2 w_2(X)\ .$$

We refer the interested reader to [MS74, §12] for an introductory discussion on obstruction theory on vector bundles.

2.2 The exterior differential and the Nijenhuis tensor

Let us now explore the geometry of almost complex manifolds. For the remainder of the section (X^n, J) will denote an almost complex manifold of (complex) dimension n.

Lemma 2.22. *The complexified tangent bundle $TX_{\mathbb{C}} \coloneqq TX \otimes_{\mathbb{R}} \mathbb{C}$ splits as a direct sum of complex bundles $TX^{1,0} \oplus TX^{0,1}$ of complex dimension n, given by*

$$TX^{1,0} = \ker\left(i\,\mathrm{Id} - J\right) \qquad\qquad TX^{0,1} = \ker\left(i\,\mathrm{Id} + J\right)$$

Proof. The minimal polynomial of J is $x^2 - 1 = (x - i)(x + i)$, which means J is diagonalisable over $\mathbb{C}$. The bundles $TX^{1,0}$ and $TX^{0,1}$ are the corresponding eigenbundles □

Remark 2.23. While $TX^{1,0}$ and $TX^{0,1}$ are not in general isomorphic as complex bundles, they are always isomorphic as real bundles, with the isomorphism given by conjugation.

The decomposition of the complexified tangent bundle into holomorphic and anti-holomorphic parts trickles down into all associated vector bundles. In particular, we have the following decomposition of exterior k-forms:

$$\bigwedge^k T^*M \otimes \mathbb{C} = \bigoplus_{p+q=k} \bigwedge^{p,q} T^*M \qquad\qquad \bigwedge^{p,q} T^*M \coloneqq \bigwedge^p \left(T^*X^{1,0}\right) \otimes \bigwedge^q \left(T^*X^{1,0}\right)\ .$$

We denote the space of smooth sections of $\bigwedge^{p,q} T^*M$ by $\mathcal{A}^{p,q} = \Gamma(X, \bigwedge^{p,q} T^*M)$.

There is a more abstract way of understanding this decomposition. An almost complex carry a reductio of the structure group $\mathrm{GL}(n, \mathbb{C}) \subset \mathrm{GL}(2n, \mathbb{R})$, and the decomposition of k- forms into (p, q)-forms corresponds to decomposition of $\Lambda^k(\mathbb{R}^{2n})^* \otimes_{\mathbb{R}} \mathbb{C}$ into irreducible representations of $\mathrm{GL}(n, \mathbb{C})$.

We can study how the exterior differential behaves with respect to this decomposition. We have the following:

Proposition 2.24. *There exists operators $\partial : \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q}$ and $\mu : \mathcal{A}^{p,q} \to \mathcal{A}^{p+2,q-1}$ such that the exterior differential d decomposes as*

$$d = \mu + \partial + \overline{\partial} + \overline{\mu} \ ,$$

with $\overline{\partial}$ and $\overline{\mu}$ are the conjugate operators to ∂ and μ respectively.

Proof. The exterior differential d is a local operator. Any (p, q)-form γ can be written down locally as

$$\gamma = \sum_{|I|=p,|J|=q} f_{I,J} \ \alpha^I \wedge \overline{\alpha}^J$$

with $\{\alpha_1, \dots, \alpha_n\}$ a local basis of $\mathcal{A}^{1,0}$. □

Lemma 2.25. *The operators ∂ and μ satisfy the following properties:*

(i) the Leibniz rule,

(ii) ∂ is $\mathbb{C}$-linear and μ is function linear, and

(iii) the following identities hold:

$$\mu\partial + \partial\mu = 0 \ , \qquad \partial^2 + \overline{\partial}\mu + \mu\overline{\partial} = 0 \ ,$$
$$\mu^2 = 0 \ , \qquad \mu\overline{\mu} + \overline{\partial}\partial + \partial\overline{\partial} + \overline{\mu}\mu = 0 \ .$$

Proof. Exercise. □

Since μ is function-linear, we can identify the operator μ acting on $(0, 1)$-forms with a tensor $N_J \in \Gamma\big(X, \mathrm{Hom}(T^*X^{0,1}, \bigwedge^2 T^*X^{1,0})\big)$ such that $\mu(\alpha) = -N_J(\alpha)$ for $\alpha \in \mathcal{A}^{0,1}$.

The tensor N_J is known as the Nijenhuis tensor and will play a key role in our discussion. Under the canonical identification $\mathrm{Hom}(T^*X^{0,1}, \bigwedge^2 T^*X^{1,0}) \cong \bigwedge^2 T^*X^{1,0} \otimes TX^{0,1}$, we can view N_J as a skew-symmetric map

$$N_J : TX^{1,0} \times TX^{1,0} \to TX^{0,1} \ .$$

Lemma 2.26. *Under the identification above, the Nijenhuis tensor is given by*

$$N_J(X, Y) = ([X, Y])^{0,1} \ .$$

Proof. Let α be a $(0,1)$-form and X, Y J-holomorphic vector fields. By the definition of μ and N_J, we have that $\big(N_J(\alpha)\big)(X,Y) = -d\alpha(X,Y)$.

Now, we can expand the right-hand side using the usual formula $d\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y])$. The terms $\alpha(X)$ and $\alpha(Y)$ by bidegree reasons, and $\alpha([X,Y])$ only depends on the $(0,1)$-part of the Lie bracket since α is a $(0,1)$-form. □

All in all, we have almost proved the following:

Proposition 2.27. *On an almost complex manifold, the following are equivalent:*

(i) $\mu = 0$*,*

(ii) The subbundle $TX^{1,0}$ *is involutive,*

(iii) $\partial^2 = 0$*.*

Proof. The equivalence between (i) and (ii) follows from Lemma 2.26. Item (i) implies (iii) by Lemma 2.25. Thus, we only need to show that (iii) implies (ii).

It suffices to show that $\overline{\partial} f([X,Y]) = 0$ for a function f and $X, Y \in TX^{1,0}$. Now, we have

$$\begin{aligned} 0 = \partial^{\,2} f(X,Y) &= (d\partial f)(X,Y) = X(\partial f(Y)) - Y(\partial f(X)) - \partial f([X,Y]) \\ &= X(df(Y)) - Y(df(X)) - \partial f([X,Y]) = df([X,Y]) - \partial f([X,Y]) \\ &= \overline{\partial} f([X,Y]) \ . \end{aligned}$$

□

An almost complex structure is called integrable if any of the above conditions is satisfied, motivated by the following computation:

Lemma 2.28. *Let* (X, J) *be a complex manifold. Then* $N_J \equiv 0$*.*

Proof. Let $\{z_1, \dots, z_n\}$ be local holomorphic coordinates. Then $\{dz_1, \dots, dz_n\}$ is (pointwise) a basis for $T^*X^{1,0}$. In particular any $\alpha \in \mathcal{A}^{1,0}$ can be locally written as

$$\alpha = \sum_{k=1}^{n} f_k dz_k \ ,$$

In particular, we have

$$d\alpha = \sum_{j,k=1}^{n} \left(\frac{\partial f_k}{\partial z_j} dz_j + \frac{\partial f_k}{\partial \overline{z_j}} dz_j \right) \wedge dz_k \ .$$

□

So the vanishing of the Nijenhuis tensor is a necessary condition for (X, J) to be a complex manifold. In fact, it is also sufficient:

Theorem 2.29 (Newlander–Nirenberg)**.** *An almost complex manifold* (X, J) *admits a compatible complex structure if and only if the almost complex structure* J *is integrable, i.e.* $N_J \equiv 0$

The proof of the Newlander–Nirenberg amounts to constructing local J-holomorphic coordinates. The details of the proof are relatively technical and involved; therefore, we will skip them. You can find a complete proof in [Dem12]

Therefore, one could define a complex manifold as a manifold equipped with an integrable almost complex structure.

Remark 2.30. In fact, one can take a more systematic approach to these questions from the point of view of G-structures. In that framework, the existence of an almost complex structure corresponds to a reduction of the frame bundle to a principal $\mathrm{GL}(n,\mathbb{C})$-bundle, the vanishing of the Nijenhuis tensor corresponds to the structure being 1-integrable, and the Newlander-Nirenberg theorem says that there are no further obstructions from being 1-integrable to being integrable.

We will (hopefully) revisit the world of G-structures when we discuss the Kähler condition in Section 7.

A straightforward application of the Newlander-Nirenberg is the following:

Corollary 2.31. *Let $\iota : Z \hookrightarrow X$ be an almost complex submanifold of a complex manifold. Then Z is a complex submanifold of X*

2.3 Cohomologies in complex manifolds

As part of our discussion, we saw that (almost) complex manifolds carry natural operators that square to 0. In particular, this allows us to consider new cohomology theories for these operators.

Remark 2.32. The case of almost complex manifolds is not particularly amenable to having a good cohomology theory since the operator μ is of order 0, so cohomology groups will contain little interesting information. However, one can take this further to produce an interesting cohomology theory, but more elaborate tools are needed to realise this; see [CW21] for further details.

From now on, we restrict ourselves to the case of complex manifolds. Recall that, since $d^2 = 0$ and $d = \partial + \overline{\partial}$ on a complex manifold, we have $\partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0$. We can define four different cohomology theories on X:

Definition 2.33. Let (X, J) be a complex manifold.

- The *Dolbeault* cohomology

$$H^{p,q}_{\overline{\partial}}(X) = \frac{\ker\left(\overline{\partial} : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(X)\right)}{\mathrm{im}\left(\overline{\partial} : \mathcal{A}^{p,q-1}(X) \to \mathcal{A}^{p,q}(X)\right)}.$$

- The *de Rham* cohomology

$$H^{k}_{dR}(X) = \frac{\ker\left(d : \mathcal{A}^{k}(X) \to \mathcal{A}^{k+1}(X)\right)}{\mathrm{im}\left(d : \mathcal{A}^{k-1}(X) \to \mathcal{A}^{k}(X)\right)}.$$

item The *Bott–Chern* cohomology

$$H^{p,q}_{BC}(X) = \left(\frac{\ker\partial \,\cap\, \ker\overline{\partial}}{\mathrm{im}\,\partial\overline{\partial}}\right)^{p,q}$$

- The *Aeppli* cohomology

$$H^{p,q}_A(X) = \left(\frac{\ker \partial\overline{\partial}}{\operatorname{im}\partial \ + \ \operatorname{im}\overline{\partial}}\right)^{p,q} .$$

These are all well-defined, and there are canonical inclusion maps between the different cohomologies, induced by inclusion and projection:

$$\begin{array}{ccccc} & & H^{p,q}_{BC}(X) & & \\ & \swarrow & \downarrow & \searrow & \\ H^{p,q}_{\partial}(X) & & H^{p+q}_{dR}(X) & & H^{p,q}_{\overline{\partial}}(X) \\ & \searrow & \downarrow & \swarrow & \\ & & H^{p,q}_{A}(X) & & \end{array}$$

where $H^{p,q}_{\partial}(X)$ are defined analogously to the Dolbeault cohomology groups, and conjugation yields the isomorphisms $H^{p,q}_{\partial} \cong \overline{H^{q,p}_{\overline{\partial}}}$.

We conclude this section by computing the Dolbeault cohomology groups $H^{p,q}_{\overline{\partial}}$ on a polydisc $D_\varepsilon \subseteq \mathbb{C}^n$, for $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$, with $\varepsilon_i = \infty$ allowed. First, we need

Lemma 2.34 (Baby $\overline{\partial}$-Poincaré Lemma)**.** *Let $U \subseteq \mathbb{C}$ be an open set containing the closed ball $\overline{B_\varepsilon}$. For any $\alpha = f d\overline{z} \in \mathcal{A}^{0,1}(U)$, the function*

$$g = \frac{1}{2\pi i}\int_{B_\varepsilon} \frac{f(w)}{w - z} dw \wedge d\overline{w}$$

satisfies $\alpha = \overline{\partial} g$ on B_ε.

Proof. Let us prove that $\alpha = \overline{\partial} g$ in a neighbourhood V of $z_0 \in B_\varepsilon$. Take ψ a bump function such that $\psi|_V \equiv 1$ and $\operatorname{supp}(\psi) \subseteq B_\varepsilon$, and consider the decomposition $f = \psi f + (1-\psi) f =: f_1 + f_2$, and the induced one for g. Let us check that g_1 is a well-defined smooth function. Since f_1 has compact support, we can extend it to the entire complex plane, and by the change of coordinates $w = z + re^{i\phi}$, we have

$$\begin{aligned} \frac{1}{2\pi i}\int_{B_\varepsilon} \frac{f_1(w)}{w-z} dw \wedge d\overline{w} &= \frac{1}{2\pi i}\int_{\mathbb{C}} f(z + re^{i\phi}) \frac{(e^{i\phi}dr + ire^{i\phi}d\phi) \wedge (e^{-i\phi}dr - ire^{-i\phi}d\phi)}{re^{i\phi}} \\ &= \frac{1}{\pi}\int_{\mathbb{C}} f(z + re^{i\phi}) e^{-i\phi} d\phi \wedge dr \ , \end{aligned}$$

which is clearly smooth in B.

All that remains is to compute $\overline{\partial} g$. Since $\frac{1}{(w-z)}$ is holomorphic in the complement of V, it follows from differentiation under the integral sign that $\overline{\partial} g_2 = 0$. For g_1, using the expression above, we have

$$\begin{aligned}
\overline{\partial} g_1 &= \frac{1}{\pi}\overline{\partial}\int_{\mathbb{C}} f(z+re^{i\phi})e^{-i\phi}d\phi\wedge dr \\
&= \frac{1}{\pi}\int_{\mathbb{C}}\left(\frac{\partial f}{\partial w}\frac{\partial(z+re^{i\phi})}{\partial\overline{z}} + \frac{\partial f}{\partial\overline{w}}\overline{\left(\frac{\partial(z+re^{i\phi})}{\partial z}\right)}\right)e^{-i\phi}d\phi\wedge dr \\
&= \frac{1}{\pi}\int_{\mathbb{C}}\frac{\partial f}{\partial\overline{w}}e^{-i\phi}d\phi\wedge dr = \frac{1}{2\pi i}\int_{B}\frac{\partial f}{\partial\overline{w}}\frac{dw\wedge d\overline{w}}{w-z} \\
&= f(z)\ ,
\end{aligned}$$

where the second line follows from the chain rule from Lemma 1.3, we undid the change of variables in the third line, and the fourth line follows by the (general) Cauchy Integral Formula, Equation (3). □

By induction on the dimension and bidegree, one shows

Lemma 2.35 ($\overline{\partial}$-Poincaré lemma)**.** *Let $U \subseteq \mathbb{C}^n$ be an open set containing the closed polydisc $\overline{D_\varepsilon}$. For $q > 0$, if $\alpha \in \mathcal{A}^{p,q}(U)$ is $\overline{\partial}$-closed, there exists $\beta \in \mathcal{A}^{p,q-1}(D_\varepsilon)$ such that $\alpha = \overline{\partial}\beta$ on the polydisc.*

Proof. See [Huy05, Prop. 1.3.8]. □

We can now prove the Dolbeault–Grothendieck lemma:

Proposition 2.36. *Let D_ε be a polydisc in $\mathbb{C}^n$. Then*

$$H^{p,q}_{\overline{\partial}}(D_\varepsilon) = \begin{cases} \textit{holomorphic (p-forms)} & q = 0\ , \\ 0 & q > 0\ . \end{cases}$$

Proof. The idea is to exhaust the polydisc D_ε by a sequence of approximating polydiscs D_{ε_i}, and show that we can choose the approximating exact terms so that they do not change inside the smaller polydisc.

If $q > 1$, the difference $\beta_i - \beta_{i-1}$ will then be $\overline{\partial}$-closed, so by the $\overline{\partial}$-Poincaré lemma, we can choose γ_i such that $\overline{\partial}\gamma = \beta_i - \beta_{i-1}$. Take ψ a bump function supported on D_{ε_i} with $\psi|_{D_{\varepsilon_i}} = 1$ and set $\hat{\beta}_{i+1} = \beta_{i+1} + \overline{\partial}(\psi\gamma)$. The sequence $\hat{\beta}_i$ has the desired properties. The case $q = 1$ follows a similar idea, where now $\overline{\partial}\gamma$ is replaced by a suitable holomorphic polynomial. Full details can be found in [Huy05, Cor. 1.3.9]. □

3 Sheaves and their cohomologies

We now introduce the language and techniques of sheaf theory. While we will not use them to their fullest extent, they are a convenient tool for presenting and proving some of our results, especially when considering cohomology and vector bundles. A more detailed discussion can be found in [Wel08] and references therein. For a more thorough and comprehensive discussion using derived functors, we refer the reader to [Har77, §3].

Definition 3.1. A *presheaf* $\mathcal{F}$ of abelian groups on a topological space X is given by:

(i) For every open set $U \subseteq X$, an abelian group $\mathcal{F}(U)$

(ii) For every inclusion $V \subseteq U$, a group morphism $\mathcal{F}(U) \to \mathcal{F}(V)$ (restriction map)

such that $r_{UU} = \mathrm{id}$ and $r_{VW} \circ r_{UV} = r_{UW}$ for $W \subseteq V \subseteq U$.

Definition 3.2. A presheaf is called a *sheaf* if for every family of sections $s_i \in \mathcal{F}(U_i)$, $i \in I$, with $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$.

Equivalently, the sequence:

$$0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \to \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is exact, where the second map is $(s_i) \mapsto (s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j})$.

We can now give a (perhaps) more intuitive definition of a stalk as a direct limit of a presheaf.

Definition 3.3. The *stalk* of a presheaf $\mathcal{F}$ at $x \in X$ is:

$$\mathcal{F}_x := \varinjlim_{x \in U} \mathcal{F}(U) = \bigcup_{x \in U} \mathcal{F}(U) / \sim$$

where $s_U \sim s_V$ if $s_U|_W = s_V|_W$ for some $x \in W \subseteq U \cap V$.

Associated with a presheaf, we have an associated topological space:

Definition 3.4. For a presheaf $\mathcal{F}$, define its *Étale* space:

$$\text{Ét}(\mathcal{F}) := \bigcup_{x \in X} \mathcal{F}_x \xrightarrow{p} X \quad \text{with} \quad p^{-1}(x) = \mathcal{F}_x$$

The sets $[U, s] = \{s_x \mid x \in U\}$ for U open and $s \in \mathcal{F}(U)$, form a basis for a topology on $\text{Ét}(\mathcal{F})$, and p is a local homeomorphism.

The *sheafification*$\mathcal{F}^+$ of a presheaf $\mathcal{F}$ is defined by:

$$\mathcal{F}^+(U) = \{s : U \to \text{Ét}(\mathcal{F}) \mid s \text{ is a continuous section}\}$$

There is a natural map $\mathcal{F}(U) \to \mathcal{F}^+(U)$ compatible with restrictions. If $\mathcal{F}$ is a sheaf, this map is an isomorphism.

An easy (but important) example is that of the constant presheaf and the locally constant sheaf:

Example 3.5. *If $\mathcal{F}^{const}$ is the constant presheaf with $\mathcal{F}^{const}(U) = A$, then:*

$$\text{Ét}(\mathcal{F}^{const}) = X \times A^{disc}, \qquad\qquad (\mathcal{F}^{const})^+ = \underline{A}$$

Given a morphism of sheaves, we can study the associated kernel and image. First, we have

Lemma 3.6. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then the presheaf $\ker \varphi$ is a sheaf.*

Proof. To prove that $\ker\varphi$ is a sheaf, we need to prove that, for Uopen and $\{U_i\}$ an open cover of U, we have

(i) (Existence) if $s_i \in \ker\varphi(U_i)$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$, then there exists $s \in \ker\varphi(U)$ such that $s|_{U_i} = s_i$ for all i;

(ii) (Uniqueness) if $s \in \ker\varphi(U)$ and $s|_{U_i} = 0$, then $s = 0$.

To show (i), notice that the candidate s exists in $\mathcal{F}(U)$ since $\mathcal{F}$ is a sheaf. Thus, we only need to show that $s \in \ker\varphi(U)$. Indeed, $\varphi(s_i) = 0$ by hypothesis, and since $\mathcal{G}$ is also a sheaf, this glue together to show that $\varphi(s) = 0$, as needed. Uniqueness follows readily since $\mathcal{F}$ is a sheaf. □

In general, however the presheaves $U \mapsto \operatorname{im}\varphi_U$ and $U \mapsto \operatorname{coker}\varphi_U$ are not sheaves. For instance, one may consider the image presheaf of the exponential map $\exp : \mathcal{O}_\mathbb{C} \to \mathcal{O}_\mathbb{C}^*$. Then, for an open set U, $\exp(U)$ is the ring of holomorphic functions on U with a well-defined logarithm. But taking $U_1 = \mathbb{C} \setminus \{x \geq 0\}$ and $U_2 = \mathbb{C} \setminus \{x \leq 0\}$ suffices to see that the image presheaf is not a sheaf, as there is no logarithm defined in $\mathbb{C} \setminus \{0\}$.

Definition 3.7. For a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves, we define:

- The *image sheaf*: $\operatorname{im}\varphi := (U \mapsto \operatorname{im}\varphi_U)^+$
- The *cokernel sheaf*: $\operatorname{coker}\varphi := (U \mapsto \operatorname{coker}\varphi_U)^+$

A sequence $\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$ is called *exact* at $\mathcal{G}$ if $\ker\psi = \operatorname{im}\varphi$.

Similarly, we say the morphism φ is *injective* if $0 \to \mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ is exact; and *surjective* if $\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \to 0$ is exact.

We have the following useful characterisation of exactness:

Lemma 3.8. *The sequence $\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$ is exact iff $\mathcal{F}_x \xrightarrow{\varphi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x$ is exact for all $x \in X$.*

Proof. Exercise. □

The following sequences are examples of exact sequences:

$$0 \to \mathcal{O}_\mathbb{C} \xrightarrow{(z-p)\cdot} \mathcal{O}_\mathbb{C} \to S_\mathbb{C}(p) \to 0$$
$$0 \to \underline{\mathbb{Z}} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$$
$$0 \to \underline{\mathbb{C}} \to \mathcal{A}^0_{X,\mathbb{C}} \xrightarrow{d} \mathcal{A}^1_{X,\mathbb{C}} \to \dots$$
$$0 \to \Omega^p_X \to \mathcal{A}^{p,0}_{X,\mathbb{C}} \xrightarrow{\bar\partial} \mathcal{A}^{p,1}_{X,\mathbb{C}} \to \cdots$$
$$0 \to \mathcal{I}_p \to \mathcal{O}_X \to S_X(p) \to 0$$

where

- $\mathcal{O}_X$ is the sheaf of holomorphic functions on X,

- $\mathcal{A}^k_{X,\mathbb{C}}$ (resp. $\mathcal{A}^{p,q}_{X,\mathbb{C}}$) is the sheaf of smooth sections of $\bigwedge^k T^*X$ (resp. $\bigwedge^{p,q} T^*X$),
- $\mathcal{I}_Y$ is the sheaf of vanishing holomorphic functions on a complex submanifold, $Y \subseteq X$

$$\mathcal{I}_Y(U) := \left\{ f \in \mathcal{O}_X(U) \;\middle|\; f|_Y = 0 \right\}$$

- $S_X(0)$ is the skyscraper sheaf, defined as

$$S_X(p)(U) = \begin{cases} \mathbb{C} & \text{if } \; p \in U \\ 0 & \text{otherwise} \end{cases} .$$

, The reader is encouraged to go through these examples in detail and verify that they are exact sequences of sheaves, as they will appear repeatedly throughout the course.

Given a continuous map $f : X \to Y$ between topological spaces, we get induced maps on sheaves on them.

Definition 3.9. Let $f : X \to Y$ a continuous map, $\mathcal{F}$ a sheaf on X and $\mathcal{G}$ a sheaf on Y.

- The *direct image sheaf* of $\mathcal{F}$ is defined as $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$ for $U \subseteq Y$.
- The *inverse image sheaf* of $\mathcal{G}$ is defined as $f^{-1}\mathcal{G}(U) = \varinjlim_{f(U) \subseteq V} \mathcal{G}(V)$, where the direct limit runs over all open subsets V of Y that contain $f(U)$.

One needs to check that the definitions are indeed well-posed, that is, the presheaves defined above are indeed sheaves, but we omit that.

The direct and inverse image sheaves satisfy some nice properties:

Lemma 3.10. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps. Then,*

- *$g_* \circ f_* = (g \circ f)_*$ an $f^{-1} \circ g^{-1} = (g \circ f)^{-1}$,*
- *f^{-1} is exact (i.e. it preserves exactness),*
- *f_* and f^{-1} are adjoint to each other:* $\mathrm{Hom}(f^{-1}\mathcal{F}, \mathcal{G}) = \mathrm{Hom}(\mathcal{F}, f_*\mathcal{G})$.

Lemma 3.11. *Consider $\iota : Z \hookrightarrow X$ a continuous embedding, and $\mathcal{F}$ a sheaf on X. Let $\mathcal{F}|_Z = \iota^{-1}\mathcal{F}$. Then,*

- *if $Z = \{x\}$ is a point, $\mathcal{F}|_Z = \mathcal{F}_x$,*
- *if Z is closed, $\mathcal{F}(Z) = \mathcal{F}|_Z(Z)$, and*
- *if Z is open, $\mathcal{F}|_Z(V) = \mathcal{F}(Z \cap V)$.*

We omit the proofs of these lemmas. Finally, for completeness, we introduce the following definitions

Definition 3.12. A *ringed space* is a pair $(X, \mathcal{R})$ where $\mathcal{R}$ is a sheaf of rings on X.

A *morphism* of ringed spaces $(X, \mathcal{R}) \to (Y, \mathcal{S})$ is a continuous map $f : X \to Y$ together with a morphism of sheaves of rings $f^{-1}\mathcal{S} \to \mathcal{R}$.

Definition 3.13. Let $(X, \mathcal{R})$ be a ringed space. A *sheaf of $\mathcal{R}$-modules* is a sheaf of abelian groups $\mathcal{M}$ with a map $\mathcal{R} \times \mathcal{M} \to \mathcal{M}$ such that $\mathcal{M}(U)$ is an $\mathcal{R}(U)$-module for all open U.

Examples of ringed spaces are smooth manifolds, with $\mathcal{R} = \mathcal{C}^\infty_X (= \mathcal{A}^0_{X,\mathbb{R}})$, and complex manifolds, with $\mathcal{R} = \mathcal{O}_X$. Examples of $\mathcal{R}$-modules are discussed in the exercises.

3.1 Sheaf cohomology

Let us now discuss the issue of exactness (or rather its failure). We saw (or rather left as an exercise) that taking stalks is an exact operation. More generally, we have

Lemma 3.14. *Let*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of sheaves. Then, for any U, we have

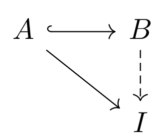
$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$$

Proof. □

In general, we lose exactness on the right, as exemplified by the fact that the exponential map $\exp : \mathcal{O}_\mathbb{C} \to \mathcal{O}^*_\mathbb{C}$ is not surjective when evaluated over $U = \mathbb{C} \setminus \{0\}$.

Cohomology is then introduced as a measure of failure for right-exactness. The correct way to understand sheaf cohomology is via the theory of derived functors, which is unfortunately beyond the scope of this course. Instead, we will present an ad-hoc construction for it.

Definition 3.15. A sheaf $\mathcal{I}$ is *injective* if for any injection $\mathcal{A} \hookrightarrow \mathcal{B}$ and map $\mathcal{A} \to \mathcal{I}$, there exists a map $\mathcal{B} \to \mathcal{I}$ making the diagram commute.



Definition 3.16. A *complex of sheaves* is a sequence:

$$\cdots \to \mathcal{F}^{i-1} \xrightarrow{d} \mathcal{F}^i \xrightarrow{d} \mathcal{F}^{i+1} \to \cdots$$

A *resolution* of a sheaf $\mathcal{F}$ is a complex $\mathcal{F}^\bullet$ with a map $\mathcal{F} \hookrightarrow \mathcal{F}^0$ that is exact. An *injective resolution* is a resolution where all $\mathcal{I}^i$ are injective.

Definition 3.17. The *sheaf cohomology* is defined as:

$$H^i(X, \mathcal{F}) := H^i\big(\Gamma(X, \mathcal{I}^\bullet)\big)$$

for an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$.

Notice that, in particular $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F}) = \mathcal{F}(X)$. A priori, this definition is subject to the existence of injective resolutions and a choice thereof. Fortunately, we have:

Proposition 3.18.

(i) *Every sheaf $\mathcal{F}$ admits an injective resolution. (The category of sheaves has enough injectives.)*

(ii) *For a morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ and injective resolutions $\mathcal{I}^\bullet$ and $\mathcal{J}^\bullet$ of $\mathcal{F}$ and $\mathcal{G}$, there exist $\varphi^k : \mathcal{I}^k \to \mathcal{J}^k$ such that*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{I}^0 & \longrightarrow & \mathcal{I}^1 & \longrightarrow & \mathcal{I}^2 & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi^0} & & \downarrow{\scriptstyle\varphi^1} & & \downarrow{\scriptstyle\varphi^2} & & \\ 0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{J}^0 & \longrightarrow & \mathcal{J}^1 & \longrightarrow & \mathcal{J}^2 & \longrightarrow & \dots \end{array}$$

commutes. Moreover, any choice of maps $\{\varphi^k\}$ induces the same maps on cohomology.

(iii) *Injective sheaves are flabby, i.e. the map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective for any $V \subseteq U$ open.*

(iv) *If*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is exact and $\mathcal{F}$ is flabby, then

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U) \to 0$$

for all open subsets U.

In particular, this implies that the sheaf cohomology groups are well-defined, and we have

Theorem 3.19. *Consider the short exact sequence of sheaves*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0 \,.$$

Then there exists a long exact sequence of cohomology:

$$0 \to H^0(X,\mathcal{F}) \to H^0(X,\mathcal{G}) \to H^0(X,\mathcal{H}) \to H^1(X,\mathcal{F}) \to H^1(X,\mathcal{G}) \to H^1(X,\mathcal{H}) \to H^2(X,\mathcal{F}) \to \dots$$

is exact

Proof. Use the fact that the injective resolution is flabby, along with the snake lemma/ diagram chasing, to construct the connecting morphisms. □

Whilst injective sheaves and injective resolutions are convenient to define sheaf cohomology, they tend to be quite cumbersome and hard to construct in explicit situations. Instead, it is more convenient to work with acyclic sheaves and resolutions

Definition 3.20. A sheaf $\mathcal{A}$ is *acyclic* if $H^i(X,\mathcal{A}) = 0$ for $i > 0$. An *acyclic resolution* is a resolution $\mathcal{A}^\bullet$ by acyclic sheaves $\mathcal{A}^i$.

The following result captures the convenience of working with acyclic resolution.

Theorem 3.21. *Let $\mathcal{A}^\bullet$ be an acyclic resolution of $\mathcal{F}$, then:*

$$H^i(X,\mathcal{F}) = H^i(\Gamma(X,\mathcal{A}^\bullet))$$

Proof. Split the resolution into short exact sequences:

$$0 \to \mathcal{K}^i \to \mathcal{A}^i \to \mathcal{K}^{i+1} \to 0$$

with $\mathcal{K}^i := \ker\left(A^i \to A^{i+1}\right) \cong \operatorname{im}\left(A^{i-1} \to A^i\right)$.. The long exact sequence of cohomology yields the desired result. □

We now claim a fact that will be of great importance, but we do not have the time to prove it:

Theorem 3.22. *All sheaves of $\mathcal{A}_{\mathbb{R}}$-modules are acyclic.*

The proof of the theorem relies on constructing a particular type of acyclic sheaves called *soft*, via a partition of unity on X. This dependence on the existence of a partition of unity is key in the construction.

As a corollary of this fact, we have

Corollary 3.23. *Let X be a smooth manifold. Then*

$$H^k_{dR}(X,\mathbb{R}) \cong H^k(X,\underline{\mathbb{R}})\ .$$

Similarly, on a complex manifold, we have

$$H^{p,q}_{\overline{\partial}}(X) \cong H^q(X,\Omega^p)$$

Proof. The smooth Poincaré lemma implies that the locally constant sheaf $\underline{\mathbb{R}}$ admits the acyclic resolution

$$\mathcal{A}^{\bullet}_{X,\mathbb{R}} := \quad 0 \to \mathcal{A}^0_{X,\mathbb{R}} \xrightarrow{d} \mathcal{A}^1_{X,\mathbb{R}} \xrightarrow{d} \mathcal{A}^2_{X,\mathbb{R}} \xrightarrow{d} \dots$$

Similarly, the $\overline{\partial}$-Poincaré lemma implies that sheaf of holomorphic p-forms admits the acyclic resolution

$$\mathcal{A}^{p,\bullet}_{X,\mathbb{R}} := \quad 0 \to \mathcal{A}^{p,0}_{X,\mathbb{R}} \xrightarrow{\overline{\partial}} \mathcal{A}^{p,1}_{X,\mathbb{R}} \xrightarrow{\overline{\partial}} \mathcal{A}^{p,2}_{X,\mathbb{R}} \xrightarrow{\overline{\partial}} \dots$$

□

3.2 Čech cohomology

We now introduce another, more combinatorial, cohomology theory for sheaves. Whilst it is more "hands-on" and computationally easy to work with, one does not have all the good properties of sheaf cohomology "on the nose".

Definition 3.24. Let $\mathcal{F}$ be a sheaf on X and $\mathcal{U} = \{U_i\}_{i\in I}$ an open cover. For each $\sigma = (i_0,\dots,i_q) \in I^{q+1}$, consider $U_\sigma = U_{i_0} \cap \dots U_{i_q}$ and $\iota_\sigma : U_\sigma \hookrightarrow X$ the inclusion.

(i) The *sheaf of Čech chains* with respect to the cover $\mathcal{U}$ is:

$$\mathcal{C}^q(\mathcal{U},\mathcal{F}) = \prod_{\sigma\in I^{q+1}} (\iota_\sigma)_*(\iota_\sigma)^{-1}\mathcal{F}$$

(ii) The *Čech boundary operator* is:

$$\begin{aligned}\delta : \mathcal{C}^q(\mathcal{U},\mathcal{F}) &\to \mathcal{C}^{q+1}(\mathcal{U},\mathcal{F}) \\ (s_\sigma)_\sigma &\mapsto \sum_{k=0}^{q+1}(-1)^k\ (s_{i_0,\dots,\check{i_k},\dots i_{q+1}})|_{U_{i_0,\dots,i_{q+1}}}\end{aligned}$$

A (tedious) computation shows that $\delta^2 = 0$, so $\left(\mathcal{C}^q(\mathcal{U},\mathcal{F}),\delta\right)$ is a complex of sheaves. In particular, we can define the (relative) Čech cohomology groups:

$$\check{H}^q(\mathcal{U},\mathcal{F}) \coloneqq \frac{\ker\left(\mathcal{C}^q(\mathcal{U},\mathcal{F}) \xrightarrow{\delta} \mathcal{C}^{q+1}(\mathcal{U},\mathcal{F})\right)}{\operatorname{im}\left(\mathcal{C}^{q-1}(\mathcal{U},\mathcal{F}) \xrightarrow{\delta} \mathcal{C}^{q}(\mathcal{U},\mathcal{F})\right)} .$$

In degree zero, we have

$$C^0(\mathcal{U},\mathcal{F}) = \prod_{U_i}\mathcal{F}(U_i) \xrightarrow{\delta} \prod_{U_i\cap U_j}\mathcal{F}(U_i\cap U_j) = C^1(\mathcal{U},\mathcal{F})$$

with $\delta(s)_{ij} = s_j|_{U_i\cap U_j} - s_i|_{U_i\cap U_j}$. Since $\mathcal{F}$ is a sheaf, $\check{H}^0(\mathcal{U},\mathcal{F}) = \ker\delta = H^0(X,\mathcal{F})$. However, the higher cohomology groups will depend on the chosen cover. To remedy this, we define

Definition 3.25. Let X be a topological space and $\mathcal{F}$ a sheaf. We define the *Čech cohomology groups* as

$$\check{H}^q(X,\mathcal{F}) = \lim_{\mathcal{U}\text{ cover}} \check{H}^q(\mathcal{U},\mathcal{F}) ,$$

where the direct limit is taken over finer and finer covers.

The result that ties up all the discussion is a celebrated result due to Leray:

Theorem 3.26 (Leray's theorem)**.** *: Let X be a smooth manifold. There is an isomorphism:*

$$H^q(X,\mathcal{F}) \cong \check{H}^q(X,\mathcal{F})$$

The main idea is to consider a good cover of X, that is, an open cover in which all open sets and all non-empty intersections of finitely many of them are contractible, and then choosing a partition of unity subordinate to this open cover.

We have been particularly vague and stated many (deep and hard) results at face value, which the reader should be pretty unhappy about (I know I am). Unfortunately, this seems to be the lesser of all evils, as proceeding in our discussion without the tools of sheaf theory and its cohomologies would prove nearly impossible. However, establishing and discussing all the material summarised in this section in detail could take an entire course on its own.

4 Meromorphic functions and Siegel's theorem

Let us put together some of the results from the previous sections. In Theorem 1.22 we saw that the stalk $\mathcal{O}_{X,x}$ (equivalently $\mathcal{O}_{\mathbb{C}^n,0}$) is an integral domain since it is a UFD, as proved in Theorem

1.22. So one may consider the corresponding field of fractions $\mathcal{K}_{X,x} := \mathrm{Quot}(\mathcal{O}_{X,x})$. Consider the following space

$$\acute{\mathrm{E}}\mathrm{t}(\mathcal{K}_X) := \bigcup_{x\in X} \mathcal{K}_x \ ,$$

with the topology induced by that of the étale space of $\mathcal{O}_X$, and define the following sheaf:

Definition 4.1. The *sheaf of meromorphic functions* on a complex manifold X is defined as

$$\mathcal{K}_X(U) = \left\{ s : U \to \acute{\mathrm{E}}\mathrm{t}(\mathcal{K}_X) \middle| \ s \text{ is continuous and } \ p \circ s = \mathrm{id}_U \right\} \ ,$$

with $p : \acute{\mathrm{E}}\mathrm{t}(\mathcal{K}_X) \to X$ the obvious projection. A *meromorphic function* is a section of this sheaf.

Note that we have chosen very suggestive notation from the start, and we are treating $\acute{\mathrm{E}}\mathrm{t}(\mathcal{K}_X)$ as an étale space, and constructed the sheaf out of it as we did for the sheafification of a presheaf.

This procedure is quite general and does not use any intrinsic properties of holomorphic functions. Indeed, this can be applied to any ringed space $(X, \mathcal{R})$ as long as the stalks of $\mathcal{R}$ are integral domains. The resulting construction is called the sheaf of rational functions.

Remark 4.2. Note that one might want to abuse notation and write $\mathcal{K}_X = \mathrm{Quot}(\mathcal{O}_X)$. However, $\mathrm{Quot}\big(\mathcal{O}_X(U)\big)$ makes no sense for any open U that is not connected, since $\mathcal{O}_X(U)$ will not be an integral domain.

Let us study the sheaf of meromorphic functions. First, notice that if X is connected, $\mathcal{K}_X(X)$ is a field, and we have an injective map of sheaves $0 \to \mathcal{O}_X \xrightarrow{\iota} \mathcal{K}_X$ with $\iota(f) = \frac{f}{1}$.

Since meromorphic functions are continuous sections $s : X \to \acute{\mathrm{E}}\mathrm{t}(\mathcal{K}_X)$, we have the following characterization:

Lemma 4.3. *Let $f \in \mathcal{K}_X(U)$. Then, for every $x \in U$ there exists an open neighbourhood V and holomorphic functions $g, h \in \mathcal{O}_X(V)$ such that the stalks g_y and h_y are coprime and $f_y = \frac{g_y}{h_y}$ for all $y \in V$. Moreover, g and h are unique, up to units in $\mathcal{O}_X(V)$.*

Proof. Combine the definition of meromorphic functions using the topology of $\acute{\mathrm{E}}\mathrm{t}(\mathcal{K}_X)$ (and thus that of $\acute{\mathrm{E}}\mathrm{t}(\mathcal{O}_X)$) with the fact that $\mathcal{O}_{X,x}$ is a UFD and Lemma 1.23. □

In particular, for any meromorphic function $f \in \mathcal{K}_X(U)$, we can define the following two analytic sets:

$$Z(f) := \left\{ x \in U \middle| \ f_x = \frac{g_x}{h_x}, \quad g(x) = 0 \right\} \qquad\qquad P(f) := \left\{ x \in U \middle| \ f_x = \frac{g_x}{h_x}, \quad h(x) = 0 \right\}$$

Notice that when f is actually holomorphic, the definition of $Z(f)$ agrees with our previous definition $Z(f) = f^{-1}(0)$. In fact, a moment's thought suffices to notice that the sets $Z(f)$ and $P(f)$ are disjoint, and that $f|_{U \setminus P(f)}$ is holomorphic.

We know that, on a compact connected complex manifold, the ring of holomorphic functions is always just $\mathbb{C}$. Let us now study how "big" the field of meromorphic functions can be. Recall from your algebra course

Definition 4.4. Let K be a field and L a field extension over K. We say that a collection $\{l_1, \dots, l_k\} \subseteq L$ is transcendentally independent if $\phi(l_1, \dots, l_k) \neq 0$ for all polynomials $\phi \in K[x_1, \dots, x_k]$. The *transcendence degree* of the extension $L|K$ is defined as

$$\operatorname{tr.deg}_K(L) = \sup_k \left\{ k \;\middle|\; \exists\, l_1, \dots l_k \text{ transcendentally independent over } K \right\}$$

We have

Theorem 4.5 (Siegel's theorem). *Let X be a compact connected complex manifold of dimension n. Then* $\operatorname{tr.deg}_{\mathbb{C}}\left(\mathcal{K}_X(X)\right) \leq n$.

Before we can prove this, we need the following lemma:

Lemma 4.6 (Schwarz lemma). *Let $\varepsilon > 0$ and $f : \overline{B_\varepsilon(0)} \to \mathbb{C}$ a holomorphic function with a zero of order k at 0. Then*

$$|f(z)| \leq C \left(\frac{|z|}{\varepsilon} \right)^k ,$$

where $C = \sup_{\partial D_\varepsilon(0)} |f|$.

Proof. Fix $0 \neq z \in D_\varepsilon(0)$ and consider the function

$$\begin{aligned} F_z : B_\varepsilon(0) &\to \mathbb{C} \\ w &\mapsto w^{-k} f\left(w \frac{z}{|z|} \right) , \end{aligned}$$

which is holomorphic since f has a zero of order k. On $\partial B_\varepsilon(0)$, we have $|F_z| \leq C$. By the maximum principle, the same bound holds for all $w \in B_\varepsilon(0)$. Thus, taking $w = |z|$, we have

$$\left| F_z\big(|z|\big) \right| = |z|^{-k} |f(z)| \leq C\varepsilon^{-k} . \qquad \square$$

Proof of Siegel's theorem. The goal is to prove that for all $f_1, \dots, f_{n+1} \in \mathcal{K}_X(X)$, there exists a polynomial $P \in K[x_1, \dots, x_{n+1}]$ such that $P(f_1, \dots, f_{n+1}) = 0$.

- <u>*Step 0*</u>: Let $z \in X$, so there exists an open neighbourhood U and $g_1, \dots, g_{n+1}, h_1, \dots, h_{n+1} \in \mathcal{O}(U)$ such that $f_i|_U = \frac{g_i}{h_i}$ for all $i \in \{1, \dots, n+1\}$. Moreover, we can assume that g_i and h_i are coprime by taking a smaller neighbourhood if necessary, by Lemma 1.23.

  Take coordinate charts around z, and consider $V_z \subseteq U$ the image of the ball of radius $1/2$ under the chosen chart. Since X is compact, we can find points $z_1, \dots, z_N$ such that the family $\{V_k\}_k$ is an open cover of X.

  For two neighbourhoods, U_k and U_l, denote $\phi_{kl,i} = \frac{g_{k,i}}{g_{l,i}} \in \mathcal{O}^*(U_k \cap U_l)$, and set

  $$C = \max_{k,l} \sup_{z \in \overline{V_k \cap V_l}} \left| \prod_{i=1}^{n+1} \phi_{kl,i}(z) \right| .$$

  Notice that the relation $\phi_{kl,i}(z)\phi_{lk,i}(z) = 1$ implies $C \geq 1$.

- *Step 1*: There exists a polynomial P of degree m such that

  $$G_k = P(f_1, \dots, f_{n+1}) \left(\prod h_{i,k}\right)^m$$

  is holomorphic on U_k and vanishes at z_k with order M for all $k \in \{1, \dots N\}$.

  Indeed, G_k is clearly holomorphic. The condition that G_k vanishes at order M is equivalent to $\partial^\alpha G_k = 0$, where $\partial^\alpha := \frac{\partial^\alpha}{\partial {z_1}^{\alpha_1} \dots \partial {z_n}^{\alpha_n}}$ is the differential operator, and $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multindex of size M.

  The collection of operators ∂^α spans a space of dimension $\binom{M-1+n}{n}$. Since the space of polynomials of degree m has dimension $\binom{m+n+1}{m}$, it suffices to choose m large enough so

  $$\binom{m+n+1}{m} > N \binom{M-1+n}{n} . \tag{6}$$

- *Step 2*: By the Schwarz Lemma 4.6, we have

  $$|G_k(z)| \leq \frac{D}{2^M}$$

  for $z \in V_k$ and $D = \max_k \sup_{z \in \overline{V_k}} |G_k(z)|$.

The goal is now equivalent to showing that $D = 0$ for an appropriate choice of m and M.

- *Step 3*: Let $z \in U_k$ such that $|G_k(z)| = D$. Thus, for some $l \in \{1, \dots, N\}$ $z \in V_l$ and so

  $$D = |G_k(z)| = |G_l(z)|\, |\phi_{kl}^m(z)| \leq \frac{C^m}{2^M} D .$$

  Thus, the constant D will vanish whenever

  $$m \log_2(C) < M . \tag{7}$$

- *Step 4*: For suitably chosen m and M, the conditions (6) and (7) can be satisfied simultaneously, so $D \equiv 0$.

  Indeed, since $\binom{m+n+1}{m}$ is a polynomial of degree $n+1$ in m and $\binom{M-1+n}{M-1}$ is a polynomial of degree n in M. Notice that this is the crucial step for which we need to take the polynomial in $n+1$ variables. □

In view of Siegel's result, we see that the following definition makes sense:

Definition 4.7. The *algebraic dimension* of a compact connected complex manifold X is

$$a(X) := \operatorname{tr.deg}_{\mathbb{C}} \left(\mathcal{K}_X(X)\right) .$$

As a first computation of the algebraic dimension, we have:

Proposition 4.8. *For all $n \in \mathbb{N}$, we have $a(\mathbb{CP}^n) = n$.*

Proof. By Siegel's theorem, it suffices to show that $a(\mathbb{CP}^n) \geq n$. Let $[Z_0 : \cdots : Z_n]$ denote homogeneous coordinates on $\mathbb{CP}^n$ and $\mathbb{C}(\xi_1, \ldots, \xi_n)$ the field of rational functions. The map

$$\begin{aligned} \Phi : \mathbb{C}(\xi_1, \ldots, \xi_n) &\to \mathcal{K}_{\mathbb{CP}^n}(\mathbb{CP}^n) \\ f(\xi_1, \ldots, \xi_n) &\mapsto f\left(\frac{Z_1}{Z_0}, \ldots \frac{Z_n}{Z_0}\right) \end{aligned}$$

is well-defined, so $\mathbb{C}(\xi_1, \ldots, \xi_n) \subseteq \mathcal{K}_{\mathbb{CP}^n}(\mathbb{CP}^n)$. □

This proves that the field extension $\mathcal{K}_{\mathbb{CP}^n}(\mathbb{CP}^n)|\mathbb{C}(\xi_1, \ldots, \xi_n)$ is algebraic. It is not hard to prove that, in fact, $\mathbb{C}(\xi_1, \ldots, \xi_n) \cong \mathcal{K}_{\mathbb{CP}^n}$, but we leave it as an exercise to the reader.

5 Holomorphic bundles, divisors and blow-ups

For convenience, we always assume that the complex manifolds we work with are connected, unless otherwise specified. Recall the definition of smooth real (resp. complex) vector bundles:

Definition 5.1. A real (resp. complex) *vector bundle* of rank r over a manifold X is a smooth manifold E together with a smooth projection $\pi : E \to X$ such that there exists an open cover $\{U_i\}$ of X and diffeomorphisms $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^r$ (resp. $\mathbb{C}^r$) such that:

(i) $\pi = \mathrm{pr}_1 \circ \varphi_i$ on $\pi^{-1}(U_i)$, where pr_1 denotes the projection to the first factor.

(ii) On $U_i \cap U_j$, the transition functions $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : (U_i \cap U_j) \times \mathbb{R}^r \to (U_i \cap U_j) \times \mathbb{R}^r$ are of the form $(x, v) \mapsto (x, g_{ij}(x)v)$ where $g_{ij} \in \mathcal{C}^\infty\big(U_i \cap U_j, \mathrm{GL}(r, \mathbb{R})\big)$

Therefore, one makes the analogue definition for the holomorphic case:

Definition 5.2. A *holomorphic vector bundle* of rank r on a complex manifold X is a complex manifold E together with a holomorphic projection $\pi : E \to X$ such that there exists an open cover $\{U_i\}$ of X and biholomorphic maps $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^r$ such that:

(i) $\pi = \mathrm{pr}_1 \circ \varphi_i$ on $\pi^{-1}(U_i)$

(ii) On $U_i \cap U_j$, the transition functions $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : (U_i \cap U_j) \times \mathbb{C}^r \to (U_i \cap U_j) \times \mathbb{C}^r$ are of the form $(x, v) \mapsto (x, g_{ij}(x)v)$ where $g_{ij} : U_i \cap U_j \to GL_r(\mathbb{C})$ are holomorphic.

The first obvious example of a holomorphic vector bundle is the holomorphic tangent bundle of a complex manifold:

Lemma 5.3. *For X a complex manifold, the bundle $T^{1,0}X \subseteq TX \otimes \mathbb{C}$ is holomorphic.*

Proof. Let $\{z_1, \ldots, z_n\}$ be local coordinates on the complex manifold. Then $\left\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}\right\}$ form a local basis of $T^{1,0}X$. The holomorphicity condition on a change of basis between trivialisations is precisely the condition that X is a complex manifold. □

As in the case of smooth vector bundles, any natural construction (in a category theory sense) of vector spaces gives rise to natural constructions of holomorphic vector bundles. In particular, we have:

Lemma 5.4. *Let E, F be holomorphic vector bundles. Then the following vector bundles are holomorphic:*

(i) $E \oplus F$,

(ii) $E \otimes F$,

(iii) E^*, *the dual of* E,

(iv) $\bigwedge^k E$ *for all* $k > 0$,

Moreover, let $\varphi : E \to F$ a bundle morphism. Then the bundles $\ker \varphi$ *and* $\operatorname{coker} \varphi$ *are holomorphic.*

Vector bundles are classified by the appropriate (Čech) cohomology group:

Proposition 5.5. *Up to isomorphism, we have the following correspondences:*

- *real vector bundles of rank r* $\xleftrightarrow{1:1} \check{H}^1\Big(X, \mathrm{GL}\big(r, \mathcal{C}^\infty(X, \mathbb{R})\big)\Big)$,
- *complex vector bundles of rank r* $\xleftrightarrow{1:1} \check{H}^1\Big(X, \mathrm{GL}\big(r, \mathcal{C}^\infty(X, \mathbb{C})\big)\Big)$
- *holomorphic vector bundles of rank r* $\xleftrightarrow{1:1} \check{H}^1\Big(X, \mathrm{GL}\big(r, \mathcal{O}_X\big)\Big)$,

where $\mathrm{GL}(r, \mathcal{F})$ is the sheaf of invertible rank k matrices with coefficients in the sheaf $\mathcal{F}$.

Proof. Exercise. □

Understanding and computing the groups $\check{H}^1\big(X, \mathrm{GL}(r, \mathcal{A})\big)$ is very hard, and there are no general results, except for the case $r = 1$, that we will revisit shortly.

To conclude this introduction, we introduce a generalisation of the Dolbeault operator $\overline{\partial}$ to holomorphic bundles: $\overline{\partial}_E$.

Proposition 5.6. *Let $E \to X$ be a holomorphic bundle, and let $\mathcal{A}^{p,q}_X(E)$ be the space of smooth (p,q)-forms with values in E. There exists a $\mathbb{C}$-linear operator $\overline{\partial}_E : \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q+1}(E)$ such that*

(i) it satisfies the "Leibniz" rule, i.e.

$$\overline{\partial}_E(\alpha \wedge s) = \overline{\partial}\alpha \wedge s + (-1)^{p+q} \alpha \wedge \overline{\partial}_E s$$

for $\alpha \in \mathcal{A}^{p,q}_X$ and $s \in \mathcal{A}^{p',q'}_X(E)$, and

(ii) it squares to zero, $\overline{\partial}_E^2 = 0$.

Proof. Clearly, the operator $\overline{\partial}_E$ is local. Let $\{s_1, \dots, s_k\}$ be a local holomorphic trivialisation of E, so any $s \in \mathcal{A}^{p,q}_X(E)$ is given by $\alpha_i \in \mathcal{A}^{p,q}_X$, with $s = \sum_{i=1}^k \alpha_i \otimes s_i$. We define

$$\overline{\partial}_E \alpha \coloneqq \sum_{i=1}^k \overline{\partial}(\alpha_i) \otimes s_i \ .$$

We need to check that this is well-defined. For another local trivialisation $\{t_1, \dots, t_k\}$, there exists a matrix $A = (\psi_{ij}) \in \mathrm{GL}(k, \mathcal{O}_X)$ such that $s_i = \sum_j \psi_{ji} \otimes t_j$. Thus,

$$\begin{aligned}\overline{\partial}_E s &= \sum_{i=1}^{k} \overline{\partial}(\alpha_i) \otimes s_i = \sum_{i,j=1}^{k} \overline{\partial}(\alpha_i) \otimes (\psi_{ji} \otimes t_j) \\ &= \sum_{j=1}^{k} \overline{\partial} \left(\sum_{i=1}^{k} \alpha_i \psi_{ji} \right) \otimes t_j = \overline{\partial}_E s \ ,\end{aligned}$$

where we crucially used that E is holomorphic, so the transition functions are holomorphic,. □

Conversely, we have

Theorem 5.7. *Let $E \to X$ be a complex vector bundle carrying an operator $\overline{\partial}_E$ satisfying the conditions above. Then E carries a natural holomorphic structure.*

The idea of the proof is that the integrability condition $\overline{\partial}_E^2 = 0$ acts like a "vanishing" Nijenhuis tensor, so one can adapt the Newlander-Nirenberg theorem(in fact Frobenius' theorem is enough) to produce the holomorphic local trivialising sections. We refer the reader to [Mor07, Thm. 9.2] for a direct proof using the N-N theorem, and to [DK90, Sec. 2.2] for a more general discussion.

The integrability condition $\overline{\partial}_E^2 = 0$ allows us to consider a version of Dolbeault cohomology for vector bundles:

$$H^{p,q}(X, E) \coloneqq \frac{\ker\left(\overline{\partial}_E : \mathcal{A}_X^{p,q}(E) \to \mathcal{A}_X^{p,q+1}(E) \right)}{\mathrm{im}\left(\overline{\partial}_E : \mathcal{A}_X^{p,q-1}(E) \to \mathcal{A}_X^{p,q}(E) \right)} \ .$$

Again, the $\overline{\partial}$-Poincaré lemma implies that the complex $0 \to \mathcal{A}_X^{p,0}(E) \to \mathcal{A}_X^{p,1}(E) \to \dots$ is an acyclic resolution of $E \otimes \Omega_X^p$, so

Corollary 5.8. *We have $H^{p,q}(X, E) \cong H^q(X, E \otimes \Omega_X^p)$.*

5.1 Holomorphic line bundles

Let us now focus on studying line bundles. First, as we anticipated earlier, we have

Lemma 5.9.

(i) Complex line bundles over X are in one-to-one correspondence with elements of $H^2(X, \mathbb{Z})$.

(ii) Real line bundles over X are in one-to-one correspondence with elements of $H^1(X, \mathbb{Z}/2\mathbb{Z})$.

Proof. Consider the exponential sequence

$$0 \to \underline{\mathbb{Z}} \xrightarrow{2\pi i} \mathcal{A}_{X,\mathbb{C}} \xrightarrow{\exp} \mathcal{A}_{X,\mathbb{C}}^* \to 0 \ .$$

We have a long exact sequence of cohomology

$$\cdots \to H^1(X, \mathcal{A}_{X,\mathbb{C}}) \to H^1(X, \mathcal{A}_{X,\mathbb{C}}^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{A}_{X,\mathbb{C}}) \to \dots \ . \tag{8}$$

Since $\mathcal{A}_\mathbb{C}$ is acyclic, the map $c_1 : H^1(X, \mathcal{A}^*_\mathbb{C}) \to H^2(X, \mathbb{Z})$ is a bijection. Similarly, for the real line bundle case, one considers the short exact sequence

$$0 \to \mathcal{A}_{X,\mathbb{R}} \xrightarrow{\exp} \mathcal{A}^*_{X,\mathbb{R}} \to \underline{\mathbb{Z}/2\mathbb{Z}} \to 0 \; . \qquad \square$$

Whilst $H^1\big(X, \mathrm{GL}(r, \mathcal{F})\big)$ does not carry any additional structure for $r > 1$, $H^1(X, \mathrm{GL}(1, \mathcal{F}) \cong H^1(X, \mathcal{F}^*)$ always carries the additional structure of an abelian group:

Lemma 5.10. *The set $H^1(X, \mathcal{F}^*)$ carries the structure of an abelian group, where the tensor product induces the group operation, and inverses are given by dualisation $[L]^{-1} := L^*$.*

Proof. Immediate. $\square$

Corollary 5.11. *The maps*

$$c_1 : H^1(X, \mathcal{A}^*_\mathbb{C}) \to H^2(X, \mathbb{Z}) \qquad\qquad w_1 : H^1(X, \mathcal{A}^*_\mathbb{R}) \to H^1(X, \mathbb{Z}/2\mathbb{Z})$$

are group morphisms.

The images of the maps c_1 and w_1 are called the first Chern and the first Stiefel–Whitney classes. They are the first instances of characteristic classes that were previously mentioned in Section 2.1.

Let us now focus on the case of holomorphic line bundles:

Definition 5.12. The group of isomorphism classes of line bundles is called the *Picard group*:

$$\mathrm{Pic}(X) := H^1(X, \mathcal{O}^*_X).$$

Again, by using the exponential short exact sequence, we have:

Proposition 5.13.

(i) A complex line bundle L admits a holomorphic structure if and only if $c_1(L)$ maps to zero in $H^2(X, \mathcal{O}_X)$.

(ii) The set of (non-isomorphic) holomorphic structures on a holomorphic line bundle is parametrised by $H^1(X, \mathcal{O}_X)/\,\mathrm{im}\,\big(H^1(X, \mathbb{Z})\big)$.

Proof. Comparing the smooth and holomorphic exponential sequences, we have:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{\mathbb{Z}} & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}^*_X & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \underline{\mathbb{Z}} & \longrightarrow & \mathcal{A}_\mathbb{C} & \longrightarrow & \mathcal{A}^*_\mathbb{C} & \longrightarrow & 0 \end{array}$$

The claim follows from the induced map between the two long exact sequences. $\square$

In particular, this discussion shows that, over a complex manifold with $H^2(X, \mathcal{O}_X) = 0$, any complex line bundle admits a holomorphic structure. As in the case of almost complex manifolds, this is not true for higher rank complex bundles, as we shall see.

Let us now introduce the tautological line bundle of the complex projective space $\mathbb{CP}^n$:

Proposition 5.14. *The tautological line bundle $\mathcal{O}(-1)$ on $\mathbb{P}^n$ is the line incidence variety:*

$$\mathcal{O}(-1) = \{(l, z) \mid z \in l\} \subseteq \mathbb{P}^n \times \mathbb{C}^{n+1}$$

with projection $\pi : \mathcal{O}(-1) \to \mathbb{P}^n$.

Proof. On affine charts $U_i = \{z_i \neq 0\}$, we have trivializations:

$$\pi^{-1}(U_i) \cong U_i \times \mathbb{C}, \quad (l, z) \mapsto (l, z_i)$$

The transition functions are $\psi_{ij}(l) = \frac{z_i}{z_j}$. □

Using the group structure of $\mathrm{Pic}(\mathbb{CP}^n)$, we define

Definition 5.15. For $k \in \mathbb{Z}$, define $\mathcal{O}(k) = \mathcal{O}(-1)^{\otimes(-k)}$, with$\mathcal{O}(-1)^{\otimes 0} := \mathcal{O}_{\mathbb{P}^n}$.

Notice that since $H^2(\mathbb{CP}^2, \mathbb{Z}) \cong \mathbb{Z}$ is torsion-free, all line bundles above are genuinely different, i.e. $\mathcal{O}(k) \cong \mathcal{O}(l)$ if and only if $k = l$, both in the holomorphic and complex vector bundle categories.

One can ask if $\mathcal{O}(-1)$ is a generator of $H^2(\mathbb{CP}^2, \mathbb{Z})$ and if not, what is its multiplicity. For now, we claim

Theorem 5.16. *The line bundle $\mathcal{O}(1)$ is a generator of $H^2(\mathbb{CP}^n, \mathbb{Z})$ under the first Chern class map.*

The proof requires further work, and we will postpone it until a later section. Another class of examples of line bundles that will interest us is the following:

Definition 5.17. The *canonical bundle* of a complex manifold X is the bundle of holomorphic top-degree forms $K_X = \bigwedge^{\dim X} T^*X^{1,0}$.

There is an interesting class of compact complex manifolds, characterised by their canonical bundle:

Definition 5.18. A compact complex manifold with $K_X \cong \mathcal{O}_X$ is called *(weak) Calabi–Yau.*

Remark 5.19. Some authors add the further requirement that $\bigwedge^p T^*X^{1,0}$ contains no trivial subbundles for $1 \leq p < \dim X$. If you are familiar with special holonomy, this is essentially equivalent to asking the holonomy of X to be irreducible (i.e. not locally a product).

We would like to characterise the canonical bundle of the complex projective space. We have

Theorem 5.20 (Euler Sequence)**.** *The holomorphic tangent bundle fits in the short exact sequence of sheaves:*

$$0 \to \mathcal{O}_{\mathbb{CP}^n} \to \bigoplus_{i=0}^{n} \mathcal{O}(1) \to \tau_{\mathbb{P}^n} \to 0$$

Proof. On $\mathbb{C}^{n+1}\backslash\{0\}$, take coordinates $z_0, \dots, z_n$, and $\pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{CP}^n$ the standard projection. Let $\widetilde{z}_i = \frac{z_i}{z_0}$ for $i \neq 0$ local coordinates in $\mathbb{CP}^n$. Then, we have

$$d\widetilde{z}_i\left(\pi_*\frac{\partial}{\partial z_j}\right) = d\left(\frac{z_i}{z_0}\right)\left(\frac{\partial}{\partial z_j}\right) = \frac{z_0 dz_i - z_i dz_0}{z_0^2}\left(\frac{\partial}{\partial z_i}\right) ,$$

so

$$\pi_*\left(\frac{\partial}{\partial z_i}\right) = \frac{1}{z_0}\frac{\partial}{\partial \widetilde{z}_i} \qquad \pi_*\left(\frac{\partial}{\partial z_0}\right) = -\sum_{i=1}^{n}\frac{z_i}{z_0^2}\frac{\partial}{\partial \widetilde{z}_i} .$$

Hence, for $L : \mathbb{C}^{n+1} \to \mathbb{C}$ a linear map, if we set $v_i = L \cdot \frac{\partial}{\partial z_i}$, the push forward $\pi_*(v_i)$ defines a section of $\tau_{\mathbb{CP}^n}$. In particular, $\tau_{\mathbb{CP}^n}$ is spanned $\left\{\pi_*\left(\frac{\partial}{\partial z_i}\right)\right\}$ for $i \in \{0, \dots, n\}$, with the relation

$$\sum_{i=0}^{n} z_i\frac{\partial}{\partial z_i} = 0 .$$

In particular, this implies the claim, where the maps in the short exact sequence are:

$$\begin{array}{ccccccc} 0 \to & \mathcal{O}_{\mathbb{CP}^n} \to & \bigoplus_{i=0}^{n}\mathcal{O}(1) & \to & \tau_{\mathbb{CP}^n} & \to & 0 \\ & 1 \mapsto & (z_0, \dots, z_n) & & & & \\ & & (s_0, \dots, s_n) & \mapsto & \pi_*\left(\sum_{i=0}^{n} s_i\frac{\partial}{\partial z_i}\right) & & \end{array}$$

□

In particular, by taking determinants of the Euler sequence, we have

Corollary 5.21. *The canonical bundle of* $\mathbb{P}^n$ *is* $K_{\mathbb{P}^n} = \mathcal{O}(-n-1)$.

Proof. We have

$$\det(\tau_{\mathbb{CP}^n}) \otimes \det(\mathcal{O}_{\mathbb{CP}^n}) = \det\left(\bigoplus_{i=0}^{n}\mathcal{O}(1)\right) = \mathcal{O}(n+1) ,$$

and the claim follows from the fact that $K_X = \det(\tau_X)^*$. □

5.2 Pluricanonical maps and Iitaka dimension

Given a holomorphic line bundle $L \to X$, it is a natural question to study its space of sections. Let $V = \langle s_0, \dots, s_n\rangle \subseteq H^0(X, L)$ be a linear subspace of (globally defined) holomorphic sections of L. We have the following definitions.

Definition 5.22. The *base locus* of V is the vanishing locus of sections spanning $V = \langle s_0, \dots, s_n\rangle$.

$$B_L(V) \coloneqq \left\{x \in X \middle|\ s_0(x) = \cdots = s_n(x) = 0\right\} .$$

The *pluricanonical map* of $V = \langle s_0, \dots, s_n\rangle$ is defined as

$$\begin{aligned} \phi_{s_0,\dots,s_n} : X \setminus B_L(V) &\to \mathbb{P}(V) \cong \mathbb{CP}^n \\ x &\mapsto [(\psi \circ s_0)(x) : \cdots : (\psi \circ s_n)(x)], \end{aligned}$$

for ψ a local trivialisation of L and a choice of homogeneous coordinates on $\mathbb{CP}^n$.

As usual, one routinely checks that the objects above are well-defined.

If one takes $V = H^0(X, L)$, we denote its base locus and pluricanonical map simply by B_L and ϕ_L respectively.

Proposition 5.23. *The pluricanonical map $\phi_{s_0,\ldots,s_n}$ is a well-defined holomorphic map. Moreover, for two different bases $\{s_i\}$ $\{s_i'\}$ of V, there exists a biholomorphism $\Psi : \mathbb{CP}^n \to \mathbb{CP}^n$ such that $\phi_{\{s_i\}} = \Psi \circ \phi_{\{s_i'\}}$*

This suggests that if one has a line bundle L with "enough" sections, one can hope to find a pluricanonical map such that

- it has empty base locus,
- is injective, and
- has injective differential.

Finding sufficient conditions for these conditions to be satisfied is roughly the idea behind Kodaria's embedding theorem, which we will prove in Section 8.2. However, we still have a long way to go before we can quantify what "enough" means.

For now, we introduce the concept of Iitaka dimension, as a way to measure "how many" sections a line bundle L and its powers L^k have. The group structure on the space of line bundles induces a map

$$H^0(X, L_1) \otimes H^0(X, L_2) \to H^0(X, L_1 \otimes L_2) \ .$$

Thus, we can consider the graded ring

$$R(X, L) = \bigoplus_{k \geq 0} H^0(X, L^k)$$

with the understanding that $L^0 = \mathcal{O}_X$. By the identity principle, it follows that $R(X, L)$ is an integral domain whenever X is connected. In particular, we can consider its field of fractions $Q(X, L)$. Moreover, since $R(X, L)$ is graded, we can further construct the following subfield of the field of fractions:

Definition 5.24. Let $Q^0(X, L)$ the subfield of $Q(X, L)$ that consists of elements of the form f/g with $f, g \in H^0(X, L^k)$ for some k.

The interest in $Q^0(X, L)$ is motivated by the following proposition:

Proposition 5.25. *For any line bundle $L \to X$, there is a map $Q^0(X, L) \to \mathcal{K}_X(X)$.*

Proof. Fix $k \geq 1$ and set $L' = L^k$. Consider $0 \neq s_1, s_2 \in H^0(X, L')$. We define a meromorphic function on X as follows.

Choose a trivialising cover (U_i, ψ_i). Then $\psi_i \circ s_j$ define holomorphic functions on U, and so $f_i = \frac{\psi_i \circ s_2}{\psi_i \circ s_1}$ is a locally defined meromorphic function. To see that $\{f_i\}$ defines a global meromorphic function, it suffices to show that it is independent of the choice of trivialisation. Indeed, we have

$$f_j = \frac{\psi_j \circ s_2}{\psi_j \circ s_1} = \frac{(\psi_{ij} \circ \psi_i) \circ s_2}{(\psi_{ij} \circ \psi_i) \circ s_1} = \frac{\lambda \psi_i \circ s_2}{\lambda \psi_i \circ s_1} = f_i \ ,$$

since $\psi_{ij} = \lambda \in \mathbb{C}^*$ since L' is a complex line bundle. □

Let us now define

Definition 5.26. Let X be a connected compact complex manifold and $L \to X$ a holomorphic line bundle. We define the *Iitaka dimension* as

$$\kappa(X, L) = \begin{cases} \operatorname{tr.deg}_{\mathbb{C}} \ Q(X, L) - 1 & \text{if} \quad Q(X, L) \neq \mathbb{C} \\ -\infty & \text{otherwise} \end{cases} .$$

If $L = K_X$ the canonical bundle, we write $\kappa(X, K_X) = \kappa(X)$ and call it the *Kodaira dimension.*

We have the following

Proposition 5.27. *For any line bundle $L \to X$, we have*

$$\kappa(X, L) \leq a(X) \ .$$

Proof. If $\kappa(X, L) = -\infty$, there's nothing to prove. Thus, it suffices to prove $\operatorname{tr.deg}_{\mathbb{C}} \ Q(R) - 1 = \operatorname{tr.deg}_{\mathbb{C}} \ Q^0(R)$ for any graded ring such that $Q(R) \neq \mathbb{C}$.

First, for $f_0, \ldots, f_k \in Q(R(X))$ are algebraically independent elements of degree d_i, $\frac{{f_1}^{e_1}}{{f_0}^{e_0}}, \ldots, \frac{{f_k}^{e_k}}{{f_0}^{e_0}}$ with $e_i = \prod_{i \neq j} d_j$ are algebraically independent elements of $Q^0(R)$. Conversely, given $f_1 \ldots, f_k \in Q^0(R)$ algebraically independent, and $f_0 \in Q(R) \setminus Q^0(R)$, then $f_0, \ldots f_k$ are algebraically independent in $Q(R)$. □

We conclude this section with the computation of the Iitaka dimensions of the line bundles $\mathcal{O}(k)$. First, we need the following result.

Proposition 5.28. *The global sections of $\mathcal{O}(k)$ are given by:*

$$H^0(\mathbb{P}^n, \mathcal{O}(k)) = \begin{cases} \mathbb{C}[z_0, \ldots, z_n]_k & \text{if } k \geq 0 \\ 0 & \text{if } k < 0 \end{cases} ,$$

where $\mathbb{C}[z_0, \ldots, z_n]_k$ denotes the space of homogeneous polynomials of degree k.

Proof. Let us prove it for $k \geq 0$. Recall that homogeneous polynomials of degree k are in one to one correspondence with k-linear symmetric forms F. Thus, a polynomial $P \in \mathbb{C}[z_0, \ldots, z_n]_k$ defines a linear map $\phi_P : (\mathbb{C}^{n+1})^{\otimes k} \to \mathbb{C}$, and thus a holomorphic map $s_P : \mathbb{CP}^n \times (\mathbb{C}^{n+1})^{\otimes k} \to \mathbb{CP}^n$ that is linear on each fibre. Restricting to $\mathcal{O}(-k)$, gives a section of $\mathcal{O}(k)$.

Explicitly, for $(l; x_1, \ldots, x_k) \in \mathcal{O}(-k)$, write $x_i = \lambda_i z$ for a fixed $z \in l$. Then $s_P(l; x_1, \ldots, x_k) = \left(\prod_i \lambda_i\right) P(z)$. We need to show that this map is bijective. Injectivity is clear, since if $s_P \equiv 0$, the polynomial P vanishes at every line so $P = 0$.

To prove surjectivity, let $t \in H^0(\mathbb{CP}^n, \mathcal{O}(k))$ and let s_P be another section induced by a polynomial of degree k. Consider the meromorphic function $F = \frac{t}{s_0} \in \mathcal{K}(\mathbb{CP}^n)$, and the associated meromorphic function on the punctured space $\widetilde{F} \coloneqq F \circ \pi \in \mathcal{K}(\mathbb{C}^{n+1} \setminus \{0\})$. Now, $G = P\widetilde{F}$ is a homogeneous holomorphic function on $\mathbb{C}^{n+1} \setminus \{0\}$ of degree k which extends to $\mathbb{C}^{n+1}$ by Hartogs' phenomenon. By Liouville's Theorem 1.13, G is a (homogeneous) polynomial of degree k, which clearly satisfies $G|_{\mathcal{O}(-k)} = t$, as needed.

The cases $k < 0$ follow from the fact that a holomorphic line bundle and its dual both admit global sections if and only if it is isomorphic to the trivial bundle (cf. Exercise Sheet). □

We readily have

Corollary 5.29.

$$\kappa(\mathbb{CP}^n, \mathcal{O}(k)) = \begin{cases} n & \textit{if} \quad k > 0 \\ 0 & \textit{if} \quad k = 0 \\ -\infty & \textit{if} \quad k < 0 \end{cases}$$

Notice that the proof we gave to Proposition 4.8 corresponds precisely to the statement

$$\kappa(\mathbb{CP}^n, \mathcal{O}(1)) \leq a(\mathbb{CP}^n)\ .$$

5.3 Divisors

Recall from Section 1, we defined analytic sets and the concept of irreducibility for analytic germs. We define

Definition 5.30. For X a complex manifolds, a *divisor* on X is a formal locally finite linear combination:

$$D = \sum a_i [Y_i]$$

where Y_i are irreducible analytic hypersurfaces and $a_i \in \mathbb{Z}$. The collection of divisors with its natural group structure is called the *group of divisors* and denoted $\mathrm{Div}(X)$.

In other words, $\mathrm{Div}(X)$ is the free abelian group over the collection of irreducible analytic hypersurfaces. In our case, local finiteness translates to the following condition: for all $x \in X$, there exists an open neighbourhood U such that $U \cap D$ is a finite sum.

Definition 5.31. A divisor $D = \sum a_i[Y_i]$ is called *effective*, and denoted by $D \geq 0$, if all $a_i \geq 0$.

Definition 5.32. Let Y be an irreducible hypersurface, $x \in Y$ and U an open neighbourhood in X and $f \in \mathcal{K}_X(U)$. The *order* of f along Y at x, denoted by $\mathrm{ord}_{Y,x}(f) \in \mathbb{Z}$, is defined as the unique integer such that

$$f_x = g_x^{\mathrm{ord}_{Y,x}(f)} h_x$$

in $\mathcal{K}_{X,x}$, where $g \in \mathcal{O}_{X,x}$ is irreducible, and $h \in \mathcal{O}^*_{X,x}$. The order of f along Y, denotedord$_Y(f) \in \mathbb{Z}$ is the order of f at x such that Y is irreducible at x [3].

[3] It is implicit in the definition that the order of f at Y does not depend on the chosen point. This is indeed the case, but we skip the proof. See [Huy05, Prop. 1.1.35] for further details.

Again, it follows from the good properties of the ring of germs $\mathcal{O}_{X,x}$ and Hilbert's Nullstellensatz that the order is well-defined. Moreover, it's not hard to check that it satisfies

$$\mathrm{ord}_Y(fg) = \mathrm{ord}_Y(f) + \mathrm{ord}_Y(g) \ .$$

In particular, we get a group morphism

$$\begin{aligned} \Phi : \mathcal{K}_X^*(X) &\to \mathrm{Div}(X) \\ f &\mapsto (f) := \sum \mathrm{ord}_Y(f)[Y] \end{aligned} \tag{9}$$

where the sum is taken over all irreducible hypersurfaces $Y \subseteq X$. An element in the image of Φ is called a *principal divisor*.

Proposition 5.33. *There is an isomorphism* $H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \xrightarrow{\cong} \mathrm{Div(X)}$

Proof. Elements in $H^0(X, \mathcal{K}_X^*/\mathcal{O}_X *)$ are given by a collection $\{U_i, f_i\}$, where $\{U_i\}$ is a cover of X, and $f_i \in \mathcal{K}_X^*(U_i)$ satisfying $f_i f_j^{-1} \in \mathcal{O}_X^*(U_i \cap U_j)$. Let $f = \{U_i, f_i\} \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ and Y an irreducible hypersurface. We claim that $\mathrm{ord}_Y(f)$ is well-defined.

We may assume that $Y \cap U_i \cap U_j \neq 0$, otherwise there's nothing to prove. Since $f \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X *)$, there exists $h_{ij} \in \mathcal{O}_X^*(U_i \cap U_j)$, we have

$$\mathrm{ord}_Y(f_i) = \mathrm{ord}_Y(h_{ij}) + \mathrm{ord}_Y(f_j) = \mathrm{ord}_Y(f_j) \ .$$

Conversely, let $D = \sum a_i[Y_i]$ be a divisor. Choose an open cover $\{U_j\}$ such that $Y_i \cap U_j = Z(g_{ij})$ for some irreducible $g_{ij} \in \mathcal{O}_X(U_j)$, which is unique up to units in $\mathcal{O}_X(U_j)$, and define

$$f_j = \prod_i g_{ij}^{a_i} \ .$$

Then $f_j \in \mathcal{K}_X(U_j)$ and $f_j/f_k \in \mathcal{O}_X^*(U_j \cap U_k)$ since $g_{ij}/g_{kj} \in \mathcal{O}_X^*(U_j \cap U_k)$. □

Using the identification above, we have:

Corollary 5.34. *There is an exact sequence:*

$$0 \to \mathbb{C}^* \to \mathcal{K}_X^*(X) \xrightarrow{\Phi} \mathrm{Div}(X) \to \mathrm{Pic}(X) \ .$$

In particular, to every divisor D, we have an associated line bundle $\mathcal{O}(D)$. The line bundle is trivial if and only if $D = (f)$ for some non-trivial meromorphic function f.

Proof. Take the long exact sequence of cohomology of the short exact sequence of sheaves:

$$0 \to \mathcal{O}_X^* \to \mathcal{K}_X^* \to \mathcal{K}_X^*/\mathcal{O}_X^* \to 0$$

□

In view of this exact sequence, we define

Definition 5.35. The *divisor class group* is

$$\mathrm{Cl}(X) = \mathrm{Div}(X)/\{(f) \big| \ f \text{ meromorphic}\}.$$

We would like to understand the image of the map

$$\mathcal{O} : \mathrm{Cl}(X) \hookrightarrow \mathrm{Pic}(X) \; .$$

While this map is, in general, not surjective, we have the following result:

Proposition 5.36. *There is the following line bundle–divisor correspondence:*

(i) Let $0 \neq s \in H^0(X, L)$ for a non-trivial line bundle. Then $\mathcal{O}\big(Z(s)\big) \cong L$.

(ii) Let $D \geq 0$ an effective divisor. Then, there exists $s \in H^0\big(X, \mathcal{O}(D)\big)$ such that $Z(s) = D$.

Proof.

(i) Let $L \in \mathrm{Pic}(X)$, and choose (U_i, φ_i) a trivialising cover. The divisor $Z(s)$ associated to $0 \neq s \in H^0(X, L)$ is given by $f := \{\varphi_i(s|_{U_i})\} \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. Then, the line bundle associated to $Z(s)$ corresponds to the cocycle $\{(U_i, f_i)\}$, but

$$f_i \cdot f_j^{-1} = \varphi_i(s|_{U_i \cap U_j}) \cdot (\varphi_j(s|_{U_i \cap U_j}))^{-1} = \varphi_i \circ \varphi_j^{-1}$$

(ii) Let $D \in \mathrm{Div}(X)$ be an effective divisor, represented by by $\{(U_i, f_i \in \mathcal{K}_X^*(U_i))\}$. Since D is effective, the functions f_i are holomorphic,$f_i \in \mathcal{O}(U_i)$. Since the line bundle $\mathcal{O}(D)$ defined via the the cocycle $\{(U_i \cap U_j, \psi_{ij} = f_i \cdot f_j^{-1})\} \in H^1(X, \mathcal{O}_X^*)$, the local holomorphic functions $f_i \in \mathcal{O}(U_i)$ define a global section $s \in H^0(X, \mathcal{O}(D))$, and $Z(s)|_{U_i} = Z(f_i) = D \cap U_i$, so $Z(s) = D$, as claimed. □

In particular, this implies

Corollary 5.37. *The image of the map $\mathcal{O} : \mathrm{Cl}(X) \to \mathrm{Pic}(X)$ is generated by holomorphic line bundles admitting global holomorphic sections.*

For $Y \subseteq X$ an irreducible hypersurface, any section $s \in H^0(X, \mathcal{O}(Y))$ satisfying $Z(s) = Y$ induces a map $\mathcal{O}_X \xrightarrow{\cdot s} \mathcal{O}(Y)$, and its dual map $\mathcal{O}(-Y) \to \mathcal{O}_X$.

Lemma 5.38. *For an effective divisor D, the induced map $\mathcal{O}(-D) \to \mathcal{O}_X$ is injective, with image the ideal sheaf $\mathcal{I}_D$, the ideal sheaf of D of holomorphic functions vanishing on D.*

The statement is clear if D is a smooth hypersurface. We omit the general proof (cf. [Huy05, Lemma 2.3.22]). Finally, let us consider the case where Y is smooth. Recall that, associated to a complex submanifold $Y \subseteq X$, one has the normal subbundle $\mathcal{N}_{Y|X}$, as the cokernel of the injection $\tau_Y \hookrightarrow \tau_X$. We would like to characterise the normal bundle of hypersurfaces in view of the preceding discussion. We have

Proposition 5.39. *Let $Y \subset X$ be a (smooth) hypersurface. Then:*

$$N_{Y/X} \cong \mathcal{O}(Y)|_Y.$$

Proof. It suffices to prove that the normal bundle $N_{Y|\mathbb{P}^n}$ has the same cocycle representation as $\mathcal{O}(Y)|_Y$. Let $\{U_i, \phi_i\}_i$ local coordinates such that $\phi_i(U_i \cap Y) = \{(z_1, \dots z_n \in \mathbb{C}^n \big| \; z_n = 0\}$.

Thus, the Jacobian of the transition functions will be of the form

$$J(\phi_{ij}) = \begin{pmatrix} J(\phi_{ij}|_Y) & * \\ 0 & \frac{\partial \phi_{ij}^n}{\partial z_n} \end{pmatrix}$$

where ϕ_{ij}^n is the n^{th} coordinate function of the transition function. Therefore , $\left\{ \frac{\partial \phi_{ij}^n}{\partial z_n} \circ \phi_j \right\}_j$ gives a cocycle representation of $\mathcal{N}_{Y/X}$.

Now, by definition, we have $\mathcal{O}(Y) \cong \left\{ \frac{s_i}{s_j} \right\}$, with s_i local equations defining Y on U_i. Now, for $y \in Y$, we can choose $s_i = \phi_i^n$, so

$$\begin{aligned} \frac{s_i}{s_j}(y) = \frac{\phi_i^n}{\phi_j^n}(y) &= \frac{(\phi_{ij} \cdot \phi_j)^n}{\phi_j^n}(y) \\ = \frac{\phi_{ij}^n}{z_n} \circ \phi_j^n(y) &= \frac{\partial \phi_{ij}^n}{\partial z_n} \circ \phi_j(y) \;, \end{aligned}$$

as needed. □

In particular, we have a straight application in the case of $\mathbb{CP}^n$:

Theorem 5.40 (Adjunction Formula for Hypersurfaces)**.** *If $Y \subset X$ is a smooth hypersurface, then: $K_Y \cong \big(K_X \otimes \mathcal{O}(Y)\big)|_Y$*

Proof. By taking determinants on the short exact sequence $0 \to \tau_Y \to \tau_X \to \mathcal{N}_{Y|X} \to 0$, we have $K_Y = K_X|_Y \otimes \det(\mathcal{N}_{Y|X})$, and the claim follows from the previous proposition. □

As a useful application, we get

Corollary 5.41. *Let $Y \subseteq \mathbb{CP}^n$ be a smooth hypersurface of degree d. Then $K_Y \cong \mathcal{O}(d-n-1)|_Y$.*

More generally, let Y is a complete intersection of hypersurfaces of degrees d_i, and set $d = \sum_i d_i$. Then once more $K_Y \cong \mathcal{O}(d-n-1)|_Y$.

Proof. Let us prove the hypersurface case, the complete intersection case follows by induction.

First, note that Proposition But $N_{Y|\mathbb{P}^n} = \mathcal{O}(d)|_Y$, by assumption and $K_{\mathbb{P}^n} \cong \mathcal{O}(-n-1)$ by Corollary 5.21.Hence:

$$K_Y = \mathcal{O}(-n-1)|_Y \otimes \mathcal{O}(d)|_Y = \mathcal{O}(d-n-1)|_Y \;.$$ □

In particular, we see that

Corollary 5.42. *Let $F \in \mathbb{C}[z_0, \dots, z_n]_{n+1}$ a homogeneous polynomial of degree $n+1$ with smooth vanishing locus. Then $Z(F) \subseteq \mathbb{CP}^n$ is a weak Calabi–Yau manifold.*

5.4 Blow-ups

We conclude this section by introducing a key construction in complex geometry: blow-ups. These give rise to the active field within complex and algebraic geometry, known as birational geometry. We will only discuss the fundamental construction and a few direct consequences.

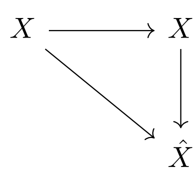
For the entire section, X will be a connected complex manifold and $Y \subset X$ a closed analytic set. The blow-up of X along Y is a triple $(\mathrm{Bl}_Y(X), E, \sigma)$, with $\hat{X} = \mathrm{Bl}_Y(X)$ a complex manifold, $E \subseteq \hat{X}$ a divisor called the exceptional divisor, and a proper holomorphic map $\sigma : \hat{X} \to X$ such that

(i) The map σ restricted to $\hat{X} \setminus \sigma^{-1}(Y)$ is a biholomorphism to $X \setminus Y$, and

(ii) The map $\sigma : \sigma^{-1}(Y) \to Y$ is biholomorphic to $\mathbb{P}(\mathcal{N}_{Y|X}) \to Y$.

The blow-up map has a characterising universal property, which we will not prove.

Theorem 5.43 (Universal property of a blow-up)**.** *Let $f : Z \to X$ a bimeromorphic map such that f restricted to $Z \setminus f^{-1}(Y)$ is holomorphic. Then there exists a unique $g : Z \to \hat{X}$ such that the diagram commutes:*



If one believes the universal characterisation of a blow-up, it is clear that the blow-up is unique, up to a unique biholomorphism. Thus, it suffices to show the existence of a blow-up, to which we will devote the rest of this section.

If one disregards the universal property, the construction of a blow-up outlined below can be taken to be the definition of *the* blow-up of X along Y. We begin by considering the example of a point.

Recall that the total space of the line bundle $\pi : \mathcal{O}(-1) \to \mathbb{CP}^n$ is the incidence variety inside $\mathbb{C}^{n+1} \times \mathbb{CP}^n$. Let us consider the other projection $\sigma : \mathcal{O}(-1) \to \mathbb{C}^{n+1}$. For $z \neq 0$ the pre-image $\sigma^{-1}(z)$ is the unique line l_z passing through $z \in \mathbb{C}^{n+1}$. However, the preimage at zero is the entire complex projective space,$\sigma^{-1}(0) = \mathbb{CP}^n$, as any line in $\mathbb{C}^{n+1}$ goes through the origin $0 \in \mathbb{C}^{n+1}$. In fact, $\sigma^{-1}(0)$ is simply the zero section of the line bundle$\pi : \mathcal{O}(-1) \to \mathbb{CP}^n$.

We define the blow-up of 0 in $\mathbb{C}^{n+1}$ as the total space of the line bundle $\big(\mathcal{O}(-1), \pi^{-1}(0), \sigma\big)$, the total space of $\mathcal{O}(-1)$, where the zero section $\pi^{-1}(0)$ is the exceptional divisor E, together with the natural projection $\sigma : \mathcal{O}(-1) \to \mathbb{C}^{n+1}$. Note that, σ is a biholomorphism away from the origin, whilst the normal bundle of 0 is simply $\mathbb{C}^{n+1}$, so $\sigma|_{\pi^{-1}(0)} : \mathbb{P}(\mathbb{C}^{n+1}) = \mathbb{CP}^n \to \{0\}$, as needed.

For an arbitrary linear subspace $\mathbb{C}^m \subseteq \mathbb{C}^{n+1}$, consider

$$Bl_m(\mathbb{C}^{n+1}) \coloneqq \big\{(z, l) \in \mathbb{C}^{n+1} \times \mathbb{CP}^{n-m} \;\big|\; z \in \langle \mathbb{C}^m, l\rangle\big\} \ .$$

Clearly, $Bl_m(\mathbb{C}^{n+1}) \to \mathbb{CP}^{n-m}$ is a $\mathbb{C}^{m+1}$-fibre bundle, and using the same argument as in the proof of Proposition 5.14, it is a holomorphic bundle, so the total space $Bl_m(\mathbb{C}^{n+1})$ is a complex manifold. The projection $\sigma : Bl_m(\mathbb{C}^{n+1}) \to \mathbb{C}^{n+1}$ gives the required blow up.

Let us construct the blow-up of a complex manifold along a submanifold $Y^m \subset X^n$. Of course, the idea is to use the previous construction as a local model and glue along different coordinate charts.

Proposition 5.44. *Let Y be a complex submanifold of X. Then the blow-up of X along Y exists.*

Proof. Take $\{U_i, \varphi_i\}$ an atlas of X such that $\varphi_i(U_i \cap Y) = \varphi(U_i) \cap \mathbb{C}^m \subseteq \mathbb{C}^n$, and consider $\sigma : \mathrm{Bl}_{\mathbb{C}^m}(\mathbb{C}^n) \to \mathbb{C}^n$ the blow-up of $\mathbb{C}^n$ along $\mathbb{C}^m$ as constructed above (note that we had $n+1$ above, rather than n).

It remains to prove that these "local" blow-ups glue along different charts, but this is easy to prove due to our "good" choice of charts. The details can be found on [Huy05, Prop. 2.5.3] □

Proposition 5.45. *The canonical bundle $K_{\hat{X}}$ of the blow-up $(\hat{X}, E, \sigma)$ is isomorphic to $\sigma^* K_X \otimes \mathcal{O}_{\hat{X}}((n-1)E)$.*

Proof. This is simply a local calculation, so it can be carried out for $X = \mathbb{C}^n$. The claim follows by computing a representing cocycle in a suitably chosen open covering of $\hat{X}$ □

In particular, we get

Corollary 5.46. *For $E = \mathbb{CP}^{n-1} \subset \hat{X} \to X$, one has $\mathcal{O}(E)|_E \cong \mathcal{O}(-1)$.*

Proof. By the previous proposition $K_{\hat{X}} \cong \sigma^* K_X \otimes \mathcal{O}\big((n-1)E\big)$, and by the adjunction formula $K_{\mathbb{CP}^{n-1}} \cong (K_{\hat{X}} \otimes \mathcal{O}(E))|_E$. Hence,$K_{\mathbb{CP}^{n-1}} \cong \mathcal{O}(nE)|_E$. Since $K_{\mathbb{CP}^{n-1}} \cong \mathcal{O}(-n)$ by Corollary 5.21 and $\mathrm{Pic}(\mathbb{CP}^{n-1}) \cong \mathbb{Z}$ is torsion free, the claim follows. □

6 Hermitian metrics and connections

Let us now combine our previous discussion with the choice of a metric on the holomorphic vector bundles. First, we go through some basic complex linear algebra results.

Definition 6.1. A *hermitian inner product* on a complex vector space E is a bilinear map $\langle \cdot, \cdot \rangle : E \otimes \overline{E} \to \mathbb{C}$ such that,

- $h(a, \overline{b}) = \overline{h(b, \overline{a})}$ (Hermitian symmetry)
- $h(a, \overline{a}) \geq 0$ with equality iff $a = 0$ (positive definiteness)

In particular, a Hermitian metric induces an anti-linear isomorphism $E \cong \overline{E}^*$. The following lemmas are standard linear algebra:

Lemma 6.2. *Let E be a complex vector space. The following objects are in one-to-one correspondence:*

(i) Hermitian inner products h,

(ii) (Real) inner products compatible with the complex structure J: $g(\cdot, \cdot) = g(J\cdot, J\cdot)$

(iii) Non-degenerate positive real $(1,1)$*-forms,* ω

Lemma 6.3. *Let* V *be an* n*-dimensional complex vector space, and* g *a compatible inner product. Consider* $\omega = g(J\ \cdot,\ \cdot)$ *the associated* $(1,1)$ *form, then* $n!\,\mathrm{vol}_g = \omega^n$*. In particular, if* $W \subseteq V$ *is a* m*-dimensional complex subspace, we have* $m!\,\mathrm{vol}_g(Y) = \omega^m$*.*

In fact, this is a characterising property, due to Wirtinger:

Lemma 6.4 (Wirtinger inequality)**.** *Let* $W^{2k} \subseteq \left(V^{2n}, g, J\right)$ *be a subspace of a Euclidean complex vector space. Denote by* ω *the associated* $(1,1)$*-form. Then*

$$\left|\omega^k\big|_W\right| \leq k!\,\mathrm{vol}_g\Big|_W ,$$

with equality if and only if W *is a complex subspace, up to orientation.*

Proof. Let $\{e_{2i-1}, e_{2i}\}$ be an orthonormal basis of W and $\{v_{2i-1}, v_{2i}\}$ its dual basis. Denote by $iota : W \hookrightarrow V$ the inclusion map. Then,

$$\iota^*\omega = \sum_{i=1}^{k} \omega(e_{2i-1}, e_{2i})\ v_{2i-1} \wedge v_{2i} .$$

Thus,

$$\left|\iota^*\left(\frac{\omega^k}{k!}\right)\right| = \prod_{i=1}^{k} \omega(e_{2i-1}, e_{2i}) vol_g = \prod_{i=1}^{k} g(Je_{2i-1}, e_{2i}) vol_g \leq vol_g ,$$

where the last inequality is simply the Cauchy-Schwarz inequality. The equality case implies $Je_{2i-1} = \pm e_{2i}$, as needed. □

On a Euclidean vector space, any form $\varphi \in \bigwedge^k V$ satisfying that $\varphi|_W \leq \mathrm{vol}_W$ for all k-planes $W \subseteq V$ is called a *(pre)calibration.*

6.1 U(n)-representation theory

Let us study the linear algebra associated with a hermitian vector space and its associated exterior algebra. Let $\left(V^{2n}, g, J\right)$ be a hermitian vector space. Recall that the space of linear maps that preserve the hermitian structure of V is a compact Lie group of dimension n^2, called the unitary group $U(n)$. The space V is naturally an irreducible $U(n)$-representation, called the standard representation. Similarly, the complexified $V_{\mathbb{C}} = V \otimes \mathbb{C}$ splits as two complex irreducible representations: $V^{1,0} \oplus V^{0,1}$ as discussed above.

We are interested in understanding how $\bigwedge^k V^*$ splits into irreducible $U(n)$-representations. First, we need the following concepts:

Definition 6.5. Let (V^{2n}, g) be a Euclidean vector space. We define the *Hodge star* operator $* : \bigwedge^k V^* \to \bigwedge^{2n-k} V^*$ by the universal property

$$\alpha \wedge *\beta = g(\alpha, \beta)\,\mathrm{vol}_g .$$

It is elementary to check that $*$ is an isometry in $\bigwedge^\bullet V^*$ satisfying $*^2 = (-1)^{k(2n-k)} = (-1)^k$ on $\bigwedge^k V^*$.

The Hodge star extends $\mathbb{C}$-linearly to $\bigwedge^\bullet V^*_{\mathbb{C}}$. With respect to the complex (p,q)-decomposition, we then have $* : \bigwedge^{p,q} \to \bigwedge^{n-q,n-p}$.

Definition 6.6. Let (V^{2n}, g, J) be a Hermitian vector space, with fundamental form ω. We define the *Lefschetz operator* by:

$$\begin{aligned} L : \bigwedge^{(p,q)} V^*_{\mathbb{C}} &\to \bigwedge^{(p+1,q+1)} V^*_{\mathbb{C}} \\ \alpha &\mapsto \alpha \wedge \omega \ . \end{aligned}$$

and its adjoint $\Lambda : \bigwedge^{(p,q)} V^*_{\mathbb{C}} \to \bigwedge^{(p-1,q-1)} V^*_{\mathbb{C}}$.

Lemma 6.7. *The adjoint of the Lefschetz operator satisfies* $\Lambda = (-1)^k * L *$.

Proof. By definition,

$$g(\Lambda\alpha, \beta)\,\mathrm{vol} = g(\alpha, L\beta)\,\mathrm{vol} = L(\beta) \wedge *\alpha = \beta \wedge \omega \wedge *\alpha = (-1)^k g\big(\beta, *\big[L(*\alpha)\big]\big)\,\mathrm{vol} \ . \qquad \square$$

We will also need the *counting operator* $H|_{\bigwedge^k V^*_{\mathbb{C}}} = (k-n)\,\mathrm{Id}$. With this in hand, we have

Proposition 6.8. *Let (V^{2n}, g, J) be a Hermitian vector space, with fundamental form ω. The Lefschetz operator satisfies*

$$[H, L] = 2L \qquad [H, \Lambda] = -2\Lambda \qquad [L, \Lambda] = H \ .$$

In particular, $\langle L, \Lambda, H\rangle$ induce an $\mathfrak{sl}(2,\mathbb{C})$-representation on $\bigwedge^\bullet V^*_{\mathbb{C}}$).

Proof. The statements $[H, L] = 2L$ and $[H, \Lambda] = -2\Lambda$ are immediate. Let us prove $[L, \Lambda] = H$ by induction over the dimension of V.

If we decompose $V = W_1 \oplus W_2$ complex subspaces, we have $\bigwedge^\bullet V^* = \bigwedge^\bullet W_1^* \otimes \bigwedge^\bullet W_2^*$ and $\omega = \omega_1 \oplus \omega_2$, so $L = \omega_1 \otimes 1 + 1 \otimes \omega_2 =: L_1 + L_2$. By linearity, it suffices to check the claim on split forms. Let $\alpha = \alpha_1 \otimes \alpha_2$ and $\beta = \beta_1 \otimes \beta_2$. Then,

$$\begin{aligned} g(\alpha, L\beta) &= g(\alpha, L_1\beta_1 \otimes \beta_2) + g(\alpha, \beta_1 \otimes L_2\beta_2) \\ &= g(\alpha_1, L_1\beta_1) g(\alpha_2, \beta_2) + g(\alpha_1, \beta_1) g(\alpha_2, L_2\beta_2) \\ &= g(\Lambda_1\alpha_1, \beta_1) g(\alpha_2, \beta_2) + g(\alpha_1, \beta_1) g(\Lambda_2\alpha_2, \beta_2) = g(\Lambda\alpha, \beta) \ . \end{aligned}$$

So $\Lambda = \Lambda_1 + \Lambda_2$. Thus, by the induction hypothesis,

$$[L, \Lambda](\alpha) = H_1(\alpha_1) \otimes \alpha_2 + \alpha_1 \otimes H_2(\alpha_2) = (k_1 - n_1)\alpha_1 \otimes \alpha_2 + (k_2 - n_2)\alpha_1 \otimes \alpha_2 = (k-n)\alpha \ .$$

Thus, the base case $n = 1$ remains. Let $\{x, y\}$ be a basis of $V \cong \mathbb{C}$ with $Jx = y$, so

$$\bigwedge^\bullet V^* = \underbrace{\bigwedge^0 V^*}_{\mathbb{R}} \oplus \underbrace{\bigwedge^1 V^*}_{\langle x, y\rangle} \oplus \underbrace{\bigwedge^2 V^*}_{\langle \omega \rangle}$$

Notice that L and Λ act trivially on $\bigwedge^1 V^*$ by degree reasons. Finally, one checks that

$$[L,\Lambda](\lambda) = -\Lambda(\lambda\omega) = -\lambda \, , \qquad\qquad [L,\Lambda](\mu\omega) = L\Lambda(\mu\omega) = \mu\omega$$

for $\lambda, \mu \in \mathbb{R}$. ☐

Corollary 6.9. *For $i \geq 1$, we have $[L^i, \Lambda](\alpha) = i(k-n+i-1)L^{i-1}(\alpha)$.*

Definition 6.10. Let (V, g, J) be a Hermitian vector space and consider the associated operators L, Λ and H. A k-form $\alpha \in \bigwedge^k V^*$ is called *primitive* if $\Lambda(\alpha) = 0$. The subspace of primitive k-forms is denoted by $P^k(V)$ or $\bigwedge_0^k V^*$.

Using Proposition 6.8, it is now easy to prove

Proposition 6.11 (Lefschetz decomposition)**.** *Let (V, g, J) be a Hermitian vector space and consider the associated operators L, Λ and H. We have an orthogonal direct sum decomposition*

$$\bigwedge^k V^* = \bigoplus_{i \geq 0} L^i(P^{k-2i}) \, .$$

Clearly, the space of primitive k-forms is a $\mathrm{U}(n)$-representation for all $k \in \{0, \ldots, n\}$. It requires some additional knowledge of representation theory to show that the primitive decompositions P^k are irreducible for all k, which we will not need. A relevant corollary of Proposition 6.11 is

Corollary 6.12. *The operator $L^{n-k} : P^k \to \Lambda^{2n-k} V^*$ is injective for all $k \leq n$.*

Finally, note that the complex structure J extends naturally to the space of k-forms: for $\alpha \in \Lambda^k V$, we define $J(\alpha)(v_1, \ldots, v_k) \coloneqq \alpha\big(Jv_1, \ldots, Jv_k\big)$. The relation between the complex structure J, the Lefschetz operator L and the Hodge star is made precise by the following results, due to Weil:

Lemma 6.13. *Let (V, g, J) be a Hermitian vector space. Consider the operators:*

- $\star = (-1)^{\binom{k}{2}+k} * J$
- $\Theta = \exp(L)\exp(-\Lambda)\exp(L)$.

Then $\star = \Theta$.

Proof. The same dimensional induction argument used in the proof of Proposition 6.8 works in this case. We leave it to the reader to verify the details.

Thus, it suffices to prove this for a complex one-dimensional space. As above, let $\{x, y\}$ be a basis of $V \cong \mathbb{C}$ with $Jx = y$, so

$$\bigwedge^\bullet V^* = \underbrace{\bigwedge^0 V^*}_{\mathbb{R}} \oplus \underbrace{\bigwedge^1 V^*}_{\langle x, y\rangle} \oplus \underbrace{\bigwedge^2 V^*}_{\langle \omega \rangle}$$

For degree 0, we have $\star 1 = (-1)^{\binom{0}{2}} * J(1) = \omega$, and

$$\begin{aligned}\Theta(1) &= \exp(L)\exp(-\Lambda)\exp(L)(1) = \exp(L)\exp(-\Lambda)(1+\omega) \\ &= \exp(L)(1+\omega-\Lambda\omega) = \exp(L)(\omega) = \omega\ .\end{aligned}$$

For degree 1, we have $\star x = - * J(x) = - * y = x$ and $\star y = - * J(y) = *x = y$, and by (bi)degree reasons, $\Theta = \mathrm{Id}$. Finally, for degree 2, we have $\star\omega = -1$ and

$$\begin{aligned}\Theta(\omega) &= \exp(L)\exp(-\Lambda)\exp(L)(\omega) = \exp(L)\exp(-\Lambda)(\omega) \\ &= \exp(L)(\omega - 1) = -1\ .\end{aligned}$$

□

As a corollary of the equality $\star = \Theta$, we get the following useful identity, known as Weil's formula:

Corollary 6.14 (Weil's formula)**.** *For all $\alpha \in P^k$, we have*

$$*L^j(\alpha) = (-1)^{\binom{k}{2}} \frac{j!}{(n-k-j)!} L^{n-k-j}\big(J(\alpha)\big)\ ,$$

Proof. Note that since $\Lambda = (-1)^k * L *$, we have $*\exp(L) = \exp(\Lambda)$ and $*\exp(\Lambda) = \exp(L)*$. Thus, for $\alpha \in P^k$, we have

$$\begin{aligned}\exp(L)\exp(-\Lambda)\exp(L)\alpha &= (-1)^{\binom{k}{2}+k} * J(\alpha) \\ \exp(\Lambda)\exp(-L) * \exp(L)\alpha &= (-1)^{\binom{k}{2}} J(\alpha) \\ * \exp(L)\alpha &= (-1)^{\binom{k}{2}} \exp(L)\exp(-\Lambda)J(\alpha) \\ &= (-1)^{\binom{k}{2}} \exp(L)J(\alpha)\ ,\end{aligned}$$

where in the last line we used that α was primitive. Expanding $\exp(L) = 1 + L + \frac{L^2}{2} + \frac{L^3}{6} + \dots$ and comparing by degree, we get Weil's formula. □

Finally, let us consider the following pairing:

Definition 6.15. Let (V, g, J) be a Hermitian vector space. For $k \le n$, the *Hodge—Riemann* pairing is defined as

$$\begin{aligned}Q : \bigwedge^k V_{\mathbb{C}}^* \times \bigwedge^k V_{\mathbb{C}}^* &\to \mathbb{C} \\ (\alpha, \beta) &\mapsto \alpha \wedge \beta \wedge \omega^{n-k}\end{aligned}$$

Proposition 6.16. *Let $\alpha \in \Lambda^{p,q} V_{\mathbb{C}}^*$ and $\beta \in \Lambda^{p',q'} V_{\mathbb{C}}^*$.*

(i) The Hodge-Riemann pairing vanishes unless $(p,q) = (q', p')$.

(ii) For $0 \neq \alpha \in P^{p,q} \subseteq \Lambda^{p,q} V_{\mathbb{C}}^*$, we have

$$Q(\alpha, \overline{\alpha}) = i^{q-p}(-1)^{\binom{k}{2}}[n-(p+q)]!\ \langle \alpha, \alpha \rangle > 0\ .$$

We leave the proof as an exercise to the reader. The definitions and discussion above carry over naturally to vector bundles:

Definition 6.17. A *hermitian metric* on a complex vector bundle E is a smooth section of $(E\otimes\overline{E})^*$ such that it induces a hermitian structure on each fibre.

By using a partition of unity subordinate to a trivialisation of E, every complex vector bundle admits a Hermitian metric, as in the Riemannian case. As usual, we have

Lemma 6.18. *If (E,h) and (F,h') are Hermitian vector bundles, then $E\otimes F$, $Hom(E,F)$, $\bigwedge^p E$ inherit natural Hermitian metrics.*

Let us consider the following example, which will play a key role in our discussion:

Example 6.19. *For a line bundle $L\to X$ with empty base locus and a basis of global sections $s_1,\dots,s_k$, we can define:*

$$h(\zeta,\xi)(x)=\frac{\langle\psi(\zeta),\psi(\xi)\rangle}{\sum_i|\psi(s_i)|^2}$$

where ψ is a local trivialization.

We are working with complex vector bundles (which are additionally holomorphic), but not all properties given by the choice of a hermitian metric extend to the complex category.

For instance, while it is true that given a short exact sequence of holomorphic vector bundles and a hermitian metric on the middle term, the sequence naturally splits in the complex bundle category, it does not generally split in the category of holomorphic bundles.

For instance, if one takes the $\mathcal{O}(-2)$-twisted Euler sequence in $\mathbb{CP}^1$:

$$0\to\mathcal{O}(-2)\to\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathcal{O}\to 0\ ,$$

then $H^0\big(\mathbb{CP}^1,\mathcal{O}(-1)^{\oplus 2}\big)=0\neq\mathbb{C}=H^0\big(\mathcal{O}\oplus\mathcal{O}(-2)\big)$.

6.2 Hodge Theory

The choice of a compatible Riemannian metric on a complex manifold X induces hermitian metrics on the bundles $\bigwedge^k T^*X$ and $\bigwedge^{p,q}T^*X$, and so the spaces $\Omega^k(M)$ and $\Omega^{p,q}(M)$ are equipped with the usual inner product and L^2-norm. As in the smooth case, one can ask:

Question 6.20. *Given a class $[\psi]\in H^{p,q}_{\overline{\partial}}(X)$, is there a representative with minimal L^2-norm?*

As expected, the answer to this question is given by the L^2-adjoint of the corresponding operator:

Proposition 6.21. *Let $\overline{\partial}^*$ be the L^2-adjoint to $\overline{\partial}$. Then ψ with $\overline{\partial}\psi=0$ has minimal L^2-norm in its $\overline{\partial}$-cohomology class if and only if $\overline{\partial}^*\psi=0$.*

Proof. First, assume we have $\psi\in\mathcal{A}^{p,q}_X$ with $\overline{\partial}\psi=0=\overline{\partial}^*\psi$. Then, for any other representative $\widetilde{\psi}=\psi+\overline{\partial}\eta$, we have

$$||\widetilde{\psi}||^2=||\psi||^2+||\overline{\partial}\eta||^2+2\,\mathrm{Re}\langle\psi,\overline{\partial}\eta\rangle=||\psi||^2+||\overline{\partial}\eta||^2+2\,\mathrm{Re}\langle\overline{\partial}^*\psi,\eta\rangle\geq||\psi||^2\ .$$

Conversely, assume ψ has minimal norm. In particular, for all $\eta \in \mathcal{A}_X^{p,q-1}$, we have $\frac{d}{dt}||\psi+t\overline{\partial}\eta||^2 = 0$. Differentiating, we have $\mathrm{Re}\langle\overline{\partial}^*\psi, \eta\rangle = 0$. By taking $\eta' = i\eta$, it follows that $\langle\overline{\partial}^*\psi, \eta\rangle = 0$. Since this holds for arbitrary η, we have $\partial^*\psi = 0$. □

The reader might have noticed that, while the statement is technically true, it requires more care than what has been put into the proof. Indeed, the L^2-adjoint of $\overline{\partial}$ is only defined on the L^2-completion of $\mathcal{A}_X^{p,q}$.

However, from Stokes' theorem, one has

Lemma 6.22. *For $\psi \in \mathcal{A}_X^{p,q}$, we have $\overline{\partial}^*\psi = -\overline{*}\overline{\partial}\overline{*}\psi = -*\partial* \in \mathcal{A}_X^{p,q-1}$, where $\overline{*}$ is the composition $\overline{*} := \bar{\cdot} \circ * : \mathcal{A}_X^{p,q} \to \mathcal{A}_X^{n-p,n-q}$*

Proof. Let $\alpha \in \mathcal{A}^{p,q}$, $\beta \in \mathcal{A}^{p,q-1}$. Then, we have

$$\langle\alpha, \overline{\partial}\beta\rangle = \int_X \overline{\partial}\beta \wedge \overline{*}\alpha = \int_X \overline{\partial}\left(\beta \wedge \overline{*}\alpha\right) + (-1)^{p+q} \int_X \beta \wedge \overline{\partial}\overline{*}\alpha = -\langle\beta, \overline{*}\overline{\partial}\overline{*}\alpha\rangle$$

where we used that $\overline{\partial}\left(\beta \wedge \overline{*}\alpha\right) = d\left(\beta \wedge \overline{*}\alpha\right)$ since $\beta \wedge \overline{*}\alpha \in \mathcal{A}_X^{n,n-1}$. □

Definition 6.23. The $\overline{\partial}$-*Laplacian operator* is defined as

$$\begin{aligned}\Delta_{\overline{\partial}} : \mathcal{A}^{p,q} &\to \mathcal{A}^{p,q}\\ \gamma &\mapsto (\overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial})\gamma \ .\end{aligned}$$

The space of $\overline{\partial}$-*harmonic forms* is $\mathcal{H}_{\overline{\partial}}^{p,q} := \ker \Delta_{\overline{\partial}} = \ker\overline{\partial} \ \cap \ker\overline{\partial}^*$.

It is straightforward to check that $\mathcal{H}^{p,q} = \ker\overline{\partial} \cap \ker\overline{\partial}^*$, so we get a map $\iota : \mathcal{H}^{p,q} \to H^{p,q}$, given by mapping each harmonic form to its cohomology class. We have

Lemma 6.24. *The map $\iota : \mathcal{H}^{p,q}(X) \to H^{p,q}(X)$ is injective.*

Proof. Let $\gamma \in \mathcal{H}^{p,q}(X)$ such that $[\gamma] = 0$, so γ is $\overline{\partial}$-exact, and there exists $\beta \in \mathcal{A}_X^{p,q-1}$ such that $\overline{\partial}\beta = \gamma$. Thus, $\overline{\partial}^*\overline{\partial}\beta = 0$, and so

$$= \langle\overline{\partial}^*\overline{\partial}\beta, \beta\rangle = ||\overline{\partial}\beta||^2 = ||\gamma|| \ .$$ □

In particular, every cohomology class has at most one harmonic representative, which is an element of minimal L^2-norm by Proposition 6.21. The fact that the map ι is surjective follows from the following key result, due to Hodge:

Theorem 6.25 (Hodge decomposition)**.** *Let X be a closed hermitian manifold. Then there exists a natural orthogonal decomposition*

$$\mathcal{A}_X^{p,q} = \overline{\partial}\mathcal{A}_X^{p,q-1} \oplus \mathcal{H}_{\overline{\partial}}^{p,q}(X) \oplus \overline{\partial}^*\mathcal{A}_X^{p,q+1} \ .$$

The spaces of $\overline{\partial}$-harmonic (p,q)- forms $\mathcal{H}^{p,q}$ are finite-dimensional.

Corollary 6.26. *The map* $\mathcal{H}^{p,q}_{\overline{\partial}}(X) \to H^{p,q}_{\overline{\partial}}$ *is an isomorphism.*

The proof of the Hodge decomposition requires some analytic tools (elliptic operators) that we omit.

We leave it to the reader (cf. Exercise 7 Sheet 6) to verify that this discussion extends naturally to holomorphic vector bundles, with respect to the $\overline{\partial}_E$ operator.

Analogously to the Poincaré duality, we have

Proposition 6.27 (Serre duality)**.** *Let X compact complex manifold and E a holomorphic vector bundle equipped with a hermitian metric h. The pairing*

$$\begin{aligned} \mathcal{A}^{p,q}(X,E) \times \mathcal{A}^{n-p,n-q}(X,E^*) &\to \mathbb{C} \\ (\alpha,\beta) &\mapsto \int_X \alpha \wedge \beta \end{aligned}$$

induces a non-degenerate pairing

$$\mathcal{H}^{p,q}_{\overline{\partial}_E}(X,E) \times \mathcal{H}^{n-p,n-q}_{\overline{\partial}_E}(X,E^*) \to \mathbb{C}$$

Proof. We give the details for the case $E = \underline{\mathbb{C}}$. If $\alpha \in \mathcal{H}^{p,q}(X)$, take $\beta = \overline{*}\alpha \in \mathcal{H}^{n-p,n-q}(X)$, so

$$\int_X \alpha \wedge \beta = \int_X \alpha \wedge \overline{*}\alpha = ||\alpha||^2 \geq 0 \ ,$$

which shows that the pairing is non-degenerate. For a general E, it suffice to take $\beta = h(\cdot,\alpha)$. $\square$

6.3 Connections

We want to extend the previous discussion on Hodge theory to the more general case of E-valued forms for a vector bundle E. If E is a holomorphic bundle, we saw that there exists a natural operator $\overline{\partial}_E$ that extends the natural $\overline{\partial}$ operator. But there is no (a priori) natural extension candidate for ∂.

To consider this extension as well as treat the more general case of complex vector bundles, we need to introduce the concept of a connection:

Definition 6.28. Consider a vector bundle $E \to X$ and $\mathcal{E}$ its associated sheaf of sections. A *connection* on E is a $\mathbb{C}$-linear map of sheaves:

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \mathcal{A}^1_X \ ,$$

satisfying the Leibniz rule:

$$\nabla(fs) = df \otimes s + f\nabla s$$

for $s \in \mathcal{E}(U)$, $f \in \mathcal{C}^\infty(U)$.

Locally, we can choose a local frame $\{s_1, \dots s_n\}$ that trivialises E. With respect to this basis, the connection ∇ is completely characterised by a collection of 1-forms ω_{ij} defined by the relation

$$\nabla s_i = \sum_j \omega_{ij} s_j \ .$$

The endomorphism-valued 1-form $\omega = (\omega_{ij}) \in \mathcal{A}^1_X(\mathrm{End}(E))$ is called the connection 1-form associated to ∇. Using the natural splitting $\mathcal{A}^1_X \cong \mathcal{A}^{1,0}_X \oplus \mathcal{A}^{1,0}_X$, we get a splitting $\nabla = \nabla^{1,0} \oplus \nabla^{0,1}$ for any connection.

Definition 6.29. Let (E, h) be a holomorphic vector bundle equipped with a hermitian metric h. A connection ∇ is called

(i) compatible if $\nabla^{0,1} = \overline{\partial}_E$,

(ii) metric if $\nabla h = 0$. That is, for any sections $s_1, s_2 \in \mathcal{E}$, we have

$$d\big(h(s_1, s_2)\big) = h(\nabla s_1, s_2) + h(s_1, \nabla s_2) \ .$$

These definitions should be reminiscent of the fundamental theorem of Riemannian geometry, where the Levi-Civita connection is characterised by being the unique torsion-free metric connection. Indeed, one has

Proposition 6.30. *Let (E, h) be a holomorphic vector bundle equipped with a hermitian metric h. There exists a unique compatible metric connection on (E, h), called the* Chern connection*.*

Proof. Let $h(s_i, s_j) = h_{ij}$ with s_i a local holomorphic frame. If ∇ exists, ω must be of type $(1, 0)$ and

$$\begin{aligned} \partial h_{ij} + \overline{\partial} h_{ij} = d(h_{ij}) &= h(\nabla s_i, s_j) + h(s_i, \nabla s_j) \\ &= h(\sum \omega_{ik} \otimes s_k, s_j) + h(s_i, \sum \omega_{jl} \otimes s_l) \\ &= \sum_k \omega_{ik} h_{kj} + \sum_l \overline{\omega}_{jl} h_{il} \end{aligned}$$

In coordinate-free notation, we have $\partial h = \omega h$ and $\overline{\partial} h = h \overline{\omega}^T$, and there is a unique solution: $\omega = \partial h \, h^{-1}$. □

Given a vector bundle E with a connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes \mathcal{A}^1_X$, we can extend it naturally to

$$\begin{aligned} \nabla : \mathcal{A}^k_X \otimes \mathcal{E} &\to \mathcal{A}^{k+1}_X \otimes \mathcal{E} \\ \alpha \otimes s &\mapsto d\alpha \otimes s + (-1)^k \alpha \wedge \nabla s \ . \end{aligned}$$

Definition 6.31. The *curvature* of a connection ∇ is the operator:

$$\nabla^2 = \nabla \circ \nabla : \mathcal{E} \to \mathcal{A}^2_X \otimes \mathcal{E}$$

Our interest in the curvature operator is motivated by the following proposition:

Proposition 6.32. *The curvature operator is function-linear. In particular, one can associate with it an element $F_\nabla \in \mathcal{A}^2_X(\mathrm{End}(E))$ such that*

$$\nabla^2 s = F_\nabla \cdot s \ .$$

The $\cdot$ represents the natural action of $\mathrm{End}(E)$ and will be omitted in the future.

Proof. Let $f \in \mathcal{A}^0_X(U)$ and $s \in \mathcal{E}(U)$ for some open U. Then

$$\nabla^2(fs) = \nabla\big(df \otimes s + f\nabla s\big) = \big(d^2 f \otimes s - df \otimes \nabla s\big) + \big(df \otimes \nabla s + f\nabla^2 s\big) = f\nabla^2 s \ . \qquad \square$$

In a local frame $\{s_i\}$, the curvature can be written down as

$$F_\nabla s_i = \nabla\nabla s_i = \nabla \sum_j \omega_{ij} s_j = \sum_{j,k} \big(d\omega_{ij} - \omega_{ik} \wedge \omega_{kj}\big) s_j \ , \tag{10}$$

In a coordinate-free set-up, we can rewrite this as

$$F_\nabla = d\omega + \frac{1}{2}[\omega, \omega] \ , \tag{11}$$

for the connection 1-form ω, where the bracket is defined as

$$[\alpha, \beta](X_1, \ldots, X_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (-1)^{|\sigma|} \left[\alpha\left(X_{\sigma(1)}, \ldots, X_{\sigma(k)}\right), \beta\left(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+l)}\right)\right] \ .$$

for $\alpha \in \mathcal{A}^k_X(\mathrm{End}(E))$ and $\beta \in \mathcal{A}^l_X(\mathrm{End}(E))$.

Equation (11) is the celebrated Maurer–Cartan formula.

Let us characterise the curvature of the Chern connection.

Lemma 6.33. *Let (E, h) be a holomorphic vector bundle equipped with a hermitian metric, and ∇ a connection on E. Then,*

(i) If ∇ is a compatible connection, $F_\nabla \in \mathcal{A}^{2,0}_X \oplus \mathcal{A}^{1,1}_X$.

(ii) If ∇ is metric, for any sections s_1, s_2, we have

$$h\big(F_\nabla(s_1), s_2\big) + h\big(s_1, F_\nabla(s_2)\big) = 0 \ .$$

(iii) The Chern connection satisfies $F_\nabla \in \mathcal{A}^{1,1}_X$.

Proof.

(i) For any connection, the splitting $\nabla = \nabla^{1,0} + \nabla^{0,1}$ yields

$$\nabla^2 = \big(\nabla^{1,0}\big)^2 + \big(\nabla^{1,0}\nabla^{0,1} + \nabla^{0,1}\nabla^{1,0}\big) + \big(\nabla^{0,1}\big)^2 \ ,$$

where $\big(\nabla^{0,1}\big)^2$ is the only $(0,2)$ component. If ∇ is compatible $\nabla^{0,1} = \overline{\partial}_E$, and the claim follows since $\overline{\partial}_E^2 = 0$ (cf. Proposition 5.6).

(ii) If ∇ is metric, we have

$$\begin{aligned} 0 = d^2 h(s_1, s_2) &= d\big(h(\nabla s_1, s_2) + h(s_1, \nabla s_2)\big) \\ &= \big[h(F_\nabla s_1, s_2) - h(\nabla s_1, \nabla s_2)\big] + \big[h(\nabla s_1, \nabla s_2) + h(s_1, F_\nabla s_2)\big] \\ &= h(F_\nabla s_1, s_2) + h(s_1, F_\nabla s_2) \ . \end{aligned}$$

(iii) Combining the two previous statements, the claim follows. □

Finally, let us say a few more words about connections for completeness. A key property of connections is the so-called Bianchi identity:

Proposition 6.34 (Bianchi identity)**.** *Let ∇ a connection on E. Then the curvature F_∇ satisfies $\nabla^{\mathrm{End}} F_\nabla = 0$, where ∇^{End} is the induced connection on the endomorphism bundle.*

Since there is little risk of confusion, in the future we will be abusing notation and using ∇ to denote the connection on E as well as the induced connection on its endomorphism bundle $\mathrm{End}(E)$.

Proof. Given the connection ∇ on E, the induced connection ∇^{End} is given by $\nabla^{\mathrm{End}}(f)(s) = \nabla\big(f(s)\big) - f\big(\nabla(s)\big)$. Using that $F_\nabla = \nabla^2$, we have

$$\big(\nabla^{\mathrm{End}} F_\nabla\big)(s) = \nabla\big(F_\nabla(s)\big) - F_\nabla\big(\nabla(s)\big) = \nabla\big(\nabla^2(s)\big) - \nabla^2\big(\nabla(s)\big) = 0 \ .$$ □

6.4 The first Chern class

Let us now revisit the first Chern class of a line bundle and giving an alternative interpretation of it.

For a line bundle L, we have $\mathrm{End}(L) = L \otimes L^* = \underline{\mathbb{C}}$, so the curvature F_∇ can be identified with a section of $\mathcal{A}^2_X$. Moreover, by the Bianchi identity, Proposition 6.34, we see that $dF_\nabla = \nabla F_\nabla = 0$, so we can consider its associated cohomology class $[F_\nabla]$. We have the following theorem:

Theorem 6.35. *Let $L \to X$ be a line bundle and ∇ a connection on it. Then*

$$[F_\nabla] = -2\pi i \; c_1(L)_{\mathbb{R}}$$

where $c_1(L) \in H^2(X, \mathbb{Z})$ is the first Chern class of L defined by the connecting map in the exponential long exact sequence, Equation (8)*, and $c_1(L)_{\mathbb{R}} = c_1(L) \otimes \mathbb{R}$. In particular, $[F_\nabla]$ is independent of the chosen connection.*

Proof. The idea is to compare the two constructions of $c_1(L)$ via its two different resolutions of the locally constant sheaf $\underline{\mathbb{C}}$:

$$
\begin{array}{ccccccc}
\underline{\mathbb{C}} & \longrightarrow & \mathcal{C}^0(\{U_i\},\underline{\mathbb{C}}) & \longrightarrow & \mathcal{C}^1(\{U_i\},\underline{\mathbb{C}}) & \longrightarrow & \mathcal{C}^2(\{U_i\},\underline{\mathbb{C}}) \\
\downarrow & & & & & & \downarrow{\scriptstyle i} \\
\mathcal{A}^0_X & & & & \mathcal{C}^1(\{U_i\},\mathcal{A}^0_X) & \xrightarrow{\delta_2} & \mathcal{C}^2(\{U_i\},\mathcal{A}^0_X) \\
\downarrow{\scriptstyle d} & & & & \downarrow{\scriptstyle d} & & \\
\mathcal{A}^1_X & & \mathcal{C}^0(\{U_i\},\mathcal{A}^1_X) & \xrightarrow{\delta_1} & \mathcal{C}^1(\{U_i\},\mathcal{A}^1_X) & & \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & & & \\
\mathcal{A}^2_X & \xrightarrow{\delta_0} & \mathcal{C}^0(\{U_i\},\mathcal{A}^2_X) & & & &
\end{array}
$$

First, let us construct a Čech cocycle describing $c_1(L)$. Let $\{(U_i,\phi_i)\}$ a trivialising cover of L with $U_{ij}=U_i\cap U_j$ simply connected. Then $[L]=[\{\phi_{ij}\}]\in H^1(X,{\mathcal{A}^1_X}^*)$.

Thus, if we set $\psi_{ij}=\log(\phi_{ij})$, we have a cocycle

$$c_{ijk}=\frac{1}{2\pi i}\big(\psi_{ij}+\psi_{jk}-\psi_{ik}\big)$$

in the locally constant integer sheaf representing $c_1(L)$.

Now for the connection construction of the first Chern class, if ∇ is a connection on L with local connection forms ω_i on U_i, we have:

$$\begin{aligned}
\omega_i &= \phi_{ij}\cdot\omega_j\cdot\phi_{ij}^{-1}+d\phi_{ij}\cdot\phi_{ij}^{-1}, \\
\omega_j-\omega_i &= -\phi_{ij}^{-1}d\phi_{ij}=-d\log\phi_{ij}.
\end{aligned}$$

Thus, putting it all together, we have

$$\begin{aligned}
\delta^0(F_\nabla) &= \{(U_i,d\omega_i)\}=d\{(U_i,\omega_i)\}\ , \\
\delta^1\big(\{(U_i,\omega_i)\}\big) &= \big\{(U_{ij},\omega_j-\omega_i)\big\}=\big\{(U_{ij},-d\log\phi_{ij})\big\}=d\big\{(U_{ij},-\log\phi_{ij})\big\}\ , \\
\delta^2\big(\{(U_{ij},-\log\phi_{ij})\}\big) &= \big\{(U_{ijk},\psi_{ik}-\psi_{ij}-\psi_{jk}\big\}=-2\pi i c_1(L)\ .
\end{aligned}$$

□

A first consequence of this theorem is that the image of $c_1^{\mathbb{R}}:\mathrm{Pic}(X)\to H^2(X,\mathbb{R})$ lies in the $H^{1,1}$-component. This gives a necessary condition for a line bundle to be holomorphic. Let us prove that it is sufficient:

Proposition 6.36. *Let $\beta\in H^{1,1}(X)\subseteq H^2(X,\mathbb{R})$ denote a complex line bundle L. Then L admits a holomorphic structure.*

Proof. Let ∇ be a connection on L. By Theorem 6.35, we know that $\big[\frac{i}{2\pi}F_\nabla\big]=\beta\in H^{1,1}(X,\mathbb{R})$. Thus, there exists a closed real $(1,1)$-form ζ such that $[\zeta]=\alpha=\big[\frac{i}{2\pi}F_\nabla\big]$.

Since $\big[\zeta-\frac{i}{2\pi}F_\nabla\big]=0$, there exists α such that $d\alpha=\zeta-\frac{1}{2\pi i}F_\nabla$.Consider the modified connection $\widetilde{\nabla}=\nabla-2\pi i\alpha$. Then $F_{\widetilde{\nabla}}=F_\nabla-2\pi i d\alpha=\zeta\in\mathcal{A}^{1,1}_X$.

Thus $\widetilde{\nabla}$ is a compatible connection, and so L admits a holomorphic structure by Theorem 5.7. □

Let us combine Theorem 6.35 with the line bundle–divisor correspondence. For that, we need to recall the following Poincaré map. Let $\iota : Y \to X$ be a smooth hypersurface. The *Poincaré map* $\eta_Y \in H^{2n-2}_{dR}(X,\mathbb{R})^* \cong H^2_{dR}(X,\mathbb{R})$ is given by

$$\begin{aligned} \eta_Y : H^{2n-2}_{dR}(X,\mathbb{R}) &\to \mathbb{R} \\ \gamma &\mapsto \langle \iota^*(\gamma), [Y] \rangle \ . \end{aligned}$$

This extends to a well-defined map

$$\begin{aligned} \eta : \mathrm{Div}(X) &\to H^2_{dR}(X,\mathbb{R}) \\ \sum_i a_i [Y_i] &\mapsto \sum_i a_i \eta_{Y_i} \ , \end{aligned}$$

since X is compact. We have

Theorem 6.37. *Let $L = \mathcal{O}(D)$ be the line bundle associated to a divisor $D \in \mathrm{Div}(X)$. Then $c_1(\mathcal{O}(D))_{\mathbb{R}} = \eta(D)$.*

Proof. Since c_1 is a linear map, we may assume $D = [Y]$ is an irreducible hypersurface. Choose h a metric on $L = \mathcal{O}(Y)$ and let ∇ be its Chern connection, with curvature F_∇. The claim of the theorem is equivalent to

$$\int_X F_\nabla \wedge \gamma = -2\pi i \int_Y \iota^*(\gamma) \ ,$$

for all $\gamma \in \Omega^{2n-2}_{closed}(X)$.

Let $\{U_i, \phi_i\}$ be a trivialising cover of L. On U_i, the hermitian metric $h|_{U_i}$ can be identified with a function h_i (since there is only one hermitian in $\mathbb{C}$ up to scale) such that

$$h(s(x), s(x)) = h_i(x) |\phi_i(s(x))|^2 \ .$$

The Chern connection is locally given by $\omega_i = \partial \log(h_i)$, and its curvature $F_\nabla = \overline{\partial}\partial \log(h_i)$. Now, take $s \in H^0(X, L)$ such that $Y = Z(s)$ in virtue of Proposition 5.36, and consider the tubular neighbourhood of Y, $D_\varepsilon \cong \{x \in X \big| \ h(s(x)) < \varepsilon\}$. Then

$$\begin{aligned} \int_X F_\nabla \wedge \gamma &= \lim_{\varepsilon \to 0} \int_{X \setminus D_\varepsilon} F_\nabla \wedge \gamma = \lim_{\varepsilon \to 0} \int_{X \setminus D_\varepsilon} \overline{\partial}\partial \log(h \circ s) \wedge \gamma \\ &= \frac{1}{2} \lim_{\varepsilon \to 0} \int_{X \setminus D_\varepsilon} d(\partial - \overline{\partial}) \log(h \circ s) \wedge \gamma = -\frac{1}{2} \lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon} (\partial - \overline{\partial}) \log(h \circ s) \wedge \gamma \end{aligned}$$

by Stokes' theorem, since γ is closed.

On the open cover U_i, we can compute

$$(\partial - \overline{\partial}) \log(h \circ s) = (\partial - \overline{\partial}) \log(|\phi_i|^2 h_i) = 2i \, \mathrm{Im} \left(\partial \log(\phi_i) \right) + (\partial - \overline{\partial}) h_i.$$

The rightmost term is bounded in the limit $\varepsilon \to 0$, since h (and thus h_i) is smooth. Taking coordinates so $Y = Z(z_1)$, it follows that $\partial \log(\phi_i) = \frac{dz_1}{z_1}$, it is a straightforward application of the Cauchy Integral Formula, Equation (4), to see that

$$\lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon \cap U_i} (\partial \log(\phi_i)) \wedge \gamma = 2\pi i \int_{Y \cap U_i} \alpha \ .$$

Putting everything together for U_i, we get

$$\int_{U_i} F_\nabla \wedge \gamma = -\frac{1}{2}\lim_{\varepsilon\to 0}\int_{\partial D_\varepsilon \cap U_i} (\partial - \overline{\partial})\log(h\circ s)\wedge\gamma = -i\,\mathrm{Im}\left(\lim_{\varepsilon\to 0}\int_{\partial D_\varepsilon\cap U_i}(\partial\log(\phi_i))\wedge\gamma\right)$$
$$= -2\pi i\,\mathrm{Im}\left(i\int_{Y\cap U_i}\alpha\right) = -2\pi i\int_{Y\cap U_i}\alpha\ .$$

Since it holds for an arbitrary trivialising open set U_i, the claim follows. □

It is worth noting that we assumed that Y was smooth in the proof above, but this assumption can be removed with much further effort.

In particular, this result gives a direct proof that the image of $c_1(\mathcal{O}(1))$ is a generator of $H^2(\mathbb{CP}^n,\mathbb{Z}) \cong \mathbb{Z}$, Theorem 5.16:

Proof of Theorem 5.16. Let $[H]\in H^2(\mathbb{CP}^n,\mathbb{Z})$, the Poincaré dual of the hyperplane class, which is a generator of the cohomology group. Since $\mathcal{O}(H)\cong\mathcal{O}(1)$, Theorem 6.35 concludes the proof. □

In particular, we get that $\mathrm{Pic}(\mathbb{CP}^n)\cong H^1(\mathbb{CP}^n,\mathcal{O}_{\mathbb{CP}^n})\rtimes H^2(\mathbb{CP}^n,\mathbb{Z})\cong H^1(\mathbb{CP}^n,\mathcal{O}_{\mathbb{CP}^n})\rtimes\mathbb{Z}$. In fact, we will see that $H^1(\mathbb{CP}^n,\mathcal{O}_{\mathbb{CP}^n})=0$, so $\mathrm{Pic}(\mathbb{CP}^n)\cong\mathbb{Z}$ and every line bundle admits a unique holomorphic structure.

6.5 Chern–Weil theory

In the previous section, we saw that the first Chern class of a line bundle can be realised as the cohomology class of the curvature of an arbitrary connection. This idea generalises to vector bundles of higher rank and forms the basis of Chern–Weil theory. More precisely, our next goal is to construct cohomological invariants of a complex vector bundle E using the curvature of an arbitrary connection on E. Unlike for line bundles, these invariants do not, in general, classify higher-rank bundles. To proceed, we first introduce the notion of G-invariant polynomials:

Definition 6.38. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. A homogeneous polynomial $P\in\mathbb{C}[\mathfrak{g}]_h$ is called G-invariant if, for all $A\in$ G, one has and all $x_1,\dots,x_k\in\mathfrak{g}$, one has

$$P(\mathrm{Ad}_A\,x_1,\dots,\mathrm{Ad}_A\,x_k) = P(x_1,\dots,x_k)\ ,$$

where $\mathrm{Ad}: G\to\mathfrak{g}$ is the adjoint representation map.

Note that, if we take $G=\mathbb{C}^*$, so $\mathfrak{g}=\mathbb{C}$, all polynomials are invariant polynomial, since $\mathbb{C}$ is a (commutative) field. As usual, by differentiation, one has

Lemma 6.39. *Let $P\in\mathbb{C}[\mathfrak{g}]_h$ be a homogeneous polynomial of degree k and assume G is connected. The polynomial P is G-invariant if and only if*

$$\sum_{i=1}^{k} P(x_1,\dots,x_{i-1},[x,x_i],x_{i+1},\dots,x_k)=0,$$

for all $x,x_1,\dots,x_k\in\mathfrak{g}$.

Given a homogeneous polynomial P, we denote by $\tilde{P}$ the associated symmetric k-linear form given by $P(X) = \tilde{P}(X, \dots, X)$, i.e. the polarization of P. Clearly, $\tilde{P}$ is G-invariant if and only P is.

Let us mention some examples of invariant symmetric k-linear forms when $G = \mathrm{GL}(r, \mathbb{C})$:

- The Chern class:
$$\exp(\mathrm{Id} + tA)) = 1 + tc_1(A) + \cdots + t^r c_r(A) \ ,$$

- The Chern character:
$$\mathrm{tr}\left(\exp^{tA}\right) = \mathrm{rk}(A) + tch_1(A) + \cdots + t^k ch_k(A) + \dots \ ,$$

- The Todd class:
$$\frac{\det(tA)}{\det(1 - \exp^{tA})} = 1 + tTd_1(A) + \cdots + t^k Td_k(A) + \dots \ .$$

Given a complex vector bundle, we associate a cohomology class to each of these invariant polynomials:

Theorem 6.40 (Chern–Weil)**.** *Let $E \to X$ be a complex vector bundle of rank r, and ∇ be a connection on E with curvature $F_\nabla \in \mathcal{A}^2_X(\mathrm{End}(E))$.*

For any invariant polynomial $P \in \mathbb{C}[\mathfrak{gl}(r, \mathbb{C})]^{\mathrm{GL}(r,\mathbb{C})}$, the form $P_\nabla = \tilde{P}(F_\nabla)$ is invariant under bundle isomorphism and closed. Its de Rham cohomology class is independent of the chosen connection.

Proof. First, let $g : E \to E$ be a bundle automorphism. Then g induces an action on the connection by pullback: $\nabla \mapsto \nabla^g = g^{-1} \circ \nabla \circ g$, and its curvature transforms as $F_{\nabla^g} = (g^{-1} \circ \nabla \circ g)^2 = g^{-1} F_\nabla g$.

Since P is invariant under the adjoint action of $\mathrm{GL}(r, \mathbb{C})$, we pointwise have
$$P_{\nabla^g} = P(g^{-1} F_\nabla g, \dots, g^{-1} F_\nabla g) = P(F_\nabla, \dots, F_\nabla) = P_\nabla \ .$$

Next, we must show that $dP_\nabla = 0$. Since P is homogeneous of degree k, the exterior derivative of $\tilde{P}(F_\nabla)$ is then given by
$$d\tilde{P}(F_\nabla) = kP(dF_\nabla, F_\nabla, \dots, F_\nabla).$$

Thus, we need to compute dF_∇. The Bianchi identity for the curvature (Prop. 6.34)) together with the Maurer–Cartan formula, Equation (29), we have
$$dF_\nabla = d_\nabla F_\nabla - [\omega, F_\nabla],$$

where d_∇ is the exterior covariant derivative associated to ∇ and ω is the local connection with connection form. Thus, locally
$$d\tilde{P}(F_\nabla) = -kP([\omega, F_\nabla], F_\nabla, \dots, F_\nabla) = 0 \ ,$$

by virtue of Lemma 6.39.

Finally, we need to prove independence of the chosen connection. Let ∇_0 and ∇_1 be two connections on E. Consider the affine family of connections

$$\nabla_t = (1-t)\nabla_0 + t\nabla_1 = \nabla_0 + t\alpha, \quad t \in [0,1],$$

with $\alpha = \nabla_1 - \nabla_0 \in \mathcal{A}^1_X(\mathrm{End}(E))$.Let $F_t = F_{\nabla_t}$ be the curvature of ∇_t. We compute its derivative using the Maurer-Cartan formula,

$$\partial_t F_t = \partial_t \left(d\omega_t + \frac{1}{2}[\omega_t, \omega_t] \right) = \partial_t \left(d(\omega_t + t'\alpha) + \frac{1}{2}[\omega_t + t'\alpha, \omega_t + t'\alpha] \right) = d_{\nabla_t}\alpha.$$

Now, let us consider the form $P(F_t)$. Using the chain rule and the multilinearity of $\tilde{P}$, we have

$$\frac{d}{dt}\tilde{P}(F_t) = kP\left(\frac{dF_t}{dt}, F_t, \dots, F_t\right) = kP\left(d_{\nabla_t}\alpha, F_t, \dots, F_t\right).$$

By the Fundamental Theorem of Calculus, we have

$$\begin{aligned} P(F_1) - P(F_0) = & \int_0^1 \frac{d}{dt}\tilde{P}(F_t)\, dt = \int_0^1 kP\left(d_{\nabla_t}\alpha, F_t, \dots, F_t\right)\, dt \\ =& k\int_0^1 d\left(P\left(\alpha, F_t, \dots, F_t\right)\right)\, dt - k(k-1)\int_0^1 P\left(\alpha, d_{\nabla_t}F_t, \dots, F_t\right)\, dt \\ =& k\, d\left(\int_0^1 P\left(\alpha, F_t, \dots, F_t\right)\, dt\right), \end{aligned}$$

where we used the Bianchi identity once more, and differentiation under the integral sign.

Therefore, the difference $P(F_{\nabla_1}) - P(F_{\nabla_0})$ is exact, and so $[P(F_{\nabla_1})] = [P(F_{\nabla_0})] \in H^{2k}_{\mathrm{dR}}(X)$, as needed. □

In particular, to every complex vector bundle $E \to X$, we can associate the cohomology classes

$$\begin{aligned} c(E) &= \left[\det\left(\mathrm{Id} + \frac{i}{2\pi}F_\nabla\right)\right] \in H^\bullet(X, \mathbb{C}), \\ \mathrm{ch}(E) &= \left[\mathrm{tr}\left(\exp\left(\frac{i}{2\pi}F_\nabla\right)\right)\right] \in H^\bullet(X, \mathbb{C}) \end{aligned}$$

The first, $c(E)$ is called the *total Chern class* of E, whilst $\mathrm{ch}(E)$ is called the *Chern character* of E. Analogously, one defines the *Todd class* $\mathrm{Td}(E)$.

Lemma 6.41. *All the classes defined above have real coefficients.*

Proof. Let $E \to X$ a complex vector bundle, and choose h a hermitian metric. Then, for a connection ∇ compatible with h, one has $\overline{F_\nabla} = -F_\nabla$, so

$$\overline{c(E,\nabla)} = \overline{\det\left(\mathrm{Id} + \frac{i}{2\pi}F_\nabla\right)} = \det\left(\mathrm{Id} - \frac{i}{2\pi}\overline{F_\nabla}\right) = \det\left(\mathrm{Id} + \frac{i}{2\pi}F_\nabla\right) = c(E,\nabla).$$

The same argument carries over for $\mathrm{ch}(E)$ and $\mathrm{Td}(E)$. □

Proposition 6.42. *Let $E, F \to X$ complex vector bundles. The Chern character satisfies:*

(i) Additivity: $\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F)$

(ii) Multiplicativity: $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \wedge \mathrm{ch}(F)$

(iii) Normalisation: For a line bundle L, we have $\mathrm{ch}(L) = \exp(c_1(L))$

Proof. Let ∇_1 and ∇_2 be connections on E and F respectively.

(i) The direct sum $E \oplus F$ carries the induced connection $\nabla = \nabla_1 \oplus \nabla_2$, with curvature

$$F_\nabla = \begin{pmatrix} F_{\nabla_1} & 0 \\ 0 & F_{\nabla_2} \end{pmatrix} .$$

Therefore, we have

$$\begin{aligned} \mathrm{ch}(E \oplus F, \nabla_1 \oplus \nabla_2) &= \mathrm{tr}\left(\exp\begin{pmatrix} \frac{i}{2\pi} F_{\nabla_1} & 0 \\ 0 & \frac{i}{2\pi} F_{\nabla_2} \end{pmatrix}\right) \\ &= \mathrm{tr}\begin{pmatrix} \exp\left(\frac{i}{2\pi} F_{\nabla_1}\right) & 0 \\ 0 & \exp\left(\frac{i}{2\pi} F_{\nabla_2}\right) \end{pmatrix} \\ &= \mathrm{tr}\left(\exp\left(\frac{i}{2\pi} F_{\nabla_1}\right)\right) + \mathrm{tr}\left(\exp\left(\frac{i}{2\pi} F_{\nabla_1}\right)\right) = \mathrm{ch}(E) + \mathrm{ch}(F) . \end{aligned}$$

(ii) The tensor product $E \otimes F$ carries the induced connection $\nabla = \nabla_1 \otimes \nabla_2$, with curvature $F_{\nabla_1} \otimes \mathrm{Id}_E + \mathrm{Id}_F \otimes F_{\nabla_2}$, and so

$$\begin{aligned} \mathrm{ch}(E \otimes F, \nabla_1 \oplus \nabla_2) &= \mathrm{tr}\left[\exp\left(\frac{i}{2\pi}(F_{\nabla_1} \otimes \mathrm{Id}_E + \mathrm{Id}_F \otimes F_{\nabla_2})\right)\right] \\ &= \mathrm{tr}\left[\exp\left(\frac{i}{2\pi} F_{\nabla_1}\right) \otimes \exp\left(\frac{i}{2\pi} F_{\nabla_2}\right)\right] \\ &= \mathrm{tr}\left[\exp\left(\frac{i}{2\pi} F_{\nabla_1}\right)\right] \wedge \mathrm{tr}\left[\exp\left(\frac{i}{2\pi} F_{\nabla_2}\right)\right] \\ &= \mathrm{ch}(E, \nabla_1) \wedge \mathrm{ch}(F, \nabla_2) \end{aligned}$$

(iii) Finally, since $\mathrm{rk}(L) = 1$, the trace operator is trivial, so choosing a connection ∇, we have

$$\mathrm{ch}(L) = \left[\exp\left(\frac{i}{2\pi} F_\nabla\right)\right] = \exp\left(\left[\frac{i}{2\pi} F_\nabla\right]\right) = \exp(c_1(L)) ,$$

by Theorem 6.35. □

To continue to study Chern classes and the Chern character, we find it convenient to introduce a fundamental topological result known as the splitting principle, which allows one to carry out computations with Chern classes by reducing them to the case of line bundles. Although it is a purely topological construction, the splitting principle has important implications for the curvature-based approach to characteristic classes.

Theorem 6.43 (Splitting Principle)**.** *Let $E \to X$ be a complex vector bundle of rank r. There exists a manifold $\mathrm{Fl}(E)$ called the flag manifold of E, and a projection $\pi : \mathrm{Fl}(E) \to X$ such that:*

(i) The pullback bundle $\pi^* E$ splits as a direct sum of line bundles:

$$\pi^* E = L_1 \oplus L_2 \oplus \cdots \oplus L_r.$$

(ii) The induced map on cohomology $\pi^* : H^*(X, \mathbb{Z}) \to H^*(\mathrm{Fl}(E), \mathbb{Z})$ is injective.

Therefore, any identity among characteristic classes of E that holds after pulling back to $\mathrm{Fl}(E)$ also holds on X.

Proof (Sketch). The flag manifold $\mathrm{Fl}(E)$ is constructed as an iterated projective bundle. First, let $\pi_1 : \mathbb{P}(E) \to X$ be the projectivisation of E. On $\mathbb{P}(E)$, we have an exact sequence

$$0 \to \mathcal{O}(-1) \to \pi_1^* E \to Q_1 \to 0,$$

where $\mathcal{O}(-1)$ is the tautological line bundle on $\mathbb{P}(E)$, and Q_1 is a vector bundle of rank $r-1$. Iterating the process, we have $\pi_2 : \mathbb{P}(Q_1) \to \mathbb{P}(E)$ and a splitting $\pi_2^*\, Q_1 = \mathcal{O}(-1) \oplus Q_2$, etcetera. After $(r-1)$ many steps, the process terminates. The composition $\pi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_{r-1} : \mathrm{Fl}(E) \to X$ gives the desired flag manifold.

The injectivity of π^* follows from the Leray-Hirsch theorem: the cohomology of $\mathrm{Fl}(E)$ is a free $\mathbb{Z}$-module over $H^*(X, \mathbb{Z})$, with basis given by the first Chern classes of the tautological line bundles. □

The splitting principle allows us to work formally with *Chern roots*. If $\pi^* E = \bigoplus_{i=1}^r L_i$, we set $x_i = c_1(L_i) \in H^2(\mathrm{Fl}(E), \mathbb{Z})$. Then:

$$\pi^*\left(c(E)\right) = \prod_{i=1}^r (1 + x_i) \qquad \text{and} \qquad \pi^*\left(\mathrm{ch}(E)\right) = \sum_{i=1}^r e^{x_i}$$

as formal identities in $H^*(\mathrm{Fl}(E))$. Since π^* is injective, any symmetric polynomial identity in the x_i that holds in $H^*(\mathrm{Fl}(E))$ descends to $H^*(X)$. In particular, one has

Proposition 6.44. *The total Chern class $c(E)$ and the Chern character $\mathrm{ch}(E)$ are equivalent over $\mathbb{Q}$. Moreover, one can write the Todd class explicitly in terms of the Chern classes $c_k(E)$.*

That is, for each k, there exists a universal linear combination $a_I \in \mathbb{Q}$ such that

$$\mathrm{ch}_k(E) = \sum_{i_1 + \cdots + i_r = k} a_{i_i,\ldots,i_k}\, c_{i_1}(E) \wedge \cdots \wedge c_{i_r}(E)\,,$$

and conversely for $c_k(E)$ and $\mathrm{ch_k}(E)$. Explicitly, for $k = 1, \ldots, 3$, one has

$$\mathrm{ch}_1(\mathrm{E}) = c_1(E), \tag{12a}$$

$$\mathrm{ch}_2(\mathrm{E}) = \frac{1}{2} c_1^2(E) - c_2(E), \tag{12b}$$

$$\mathrm{ch}_3(\mathrm{E}) = \frac{1}{6}\big(c_1^3(E) - 3c_1(E)c_2(E) + 3c_3(E)\big)\,. \tag{12c}$$

Similarly, for the Todd class, we have:

$$\mathrm{Td}_1(E) = c_1(E), \tag{13a}$$

$$\mathrm{Td}_2(E) = \frac{1}{12}\big(c_1^2(E) + c_2(E)\big) \tag{13b}$$

$$\mathrm{Td}_3(E) = \frac{1}{24}\big(c_1(E)c_2(E)\big)\ . \tag{13c}$$

The splitting principle has one more straightforward and remarkable consequence:

Theorem 6.45. *Let $E \to X$ be a complex vector bundle over a compact manifold. Then the Chern classes $c_k(E)$ lie in the image of $H^{2k}(X, \mathbb{Z})$ in $H^{2k}(X, \mathbb{R})$.*

Proof. By the splitting principle, this is equivalent to showing that symmetric products of x_i lie in $H^{2k}(\mathrm{Fl}(E), \mathbb{Z})$. But $x_i = c_1(L_i) \in H^2(\mathrm{Fl}(E), \mathbb{Z})$ by Theorem 6.35, and the claim follows. □

There is a more natural/axiomatic approach to Chern classes from the point of view of classifying bundles and K-theory, in a similar spirit to our construction of the first Chern class. One then shows that these abstract classes can be represented by suitable invariant polynomials of the curvature, as we have just done. This is the perspective taken by Milnor and Stasheff in their book [MS74].

7 Kähler Manifolds

We now move on to discuss an important class of complex manifolds: Kähler manifolds.

The idea behind Kähler manifolds is to have a compatible metric with the (almost) complex structure, not just as a hermitian structure, but it also satisfies some differential constraints, similar to the vanishing of the Nijenhuis tensor we saw in Section 2.2. In fact, we have

Definition 7.1. Let (X, g, J) be an almost hermitian manifold, and let ∇ denote the Levi-Civita connection of g. We say g is a *Kähler metric* if $\nabla J = 0$. A manifold equipped with a Kähler metric is called a *Kähler manifold*

The following proposition gives a more hands-on approach to Kähler metrics.

Proposition 7.2. *An almost hermitian manifold (X, g, J) is Kähler if and only if*

(i) its Nijenhuis tensor vanishes $N_J = 0$, and

(ii) the associated $(1,1)$ form ω is closed, $d\omega = 0$.

Proof (Sketch). First notice that $\nabla J = 0$ is equivalent to$\nabla\omega = 0$. Now, one can split $\nabla\omega$ into its totally antisymmetric part and its complement: $\nabla\omega = (\nabla\omega)^{as} + (\nabla\omega)^{\perp}$. Since the Levi–Civita connection is torsion-free, $(\nabla\omega)^{as} = d\omega$. With some work, one can identify $(\nabla\omega)^{\perp}$ with the Nijenhuis tensor. □

We refer the interested reader to the seminal paper of Gray and Hervella [GH80] for a detailed proof, as well as an extended discussion around the topic of Kähler metrics and their generalisations.

By the proposition above, the defining $(1,1)$-forms ω is a symplectic form. Thus, one can view a Kähler manifold as a complex manifold carrying a compatible symplectic structure.

Proposition 7.3. *Let (X, g, J) be a hermitian manifold. Then g is a Kähler metric if and only if the Levi-Civita and the Chern connections coincide.*

Proof. The necessity is clear. Sufficiency is proved in detail in [Huy05, Prop. 4. A.7]. □

Let us discuss some examples. First, $\mathbb{C}^n$ with its standard metric is a Kähler manifold. Similarly, by degree reasons, any Riemann surface is Kähler, the same way it was necessarily complex. The first non-trivial example is given by the unit disk, with $\omega = \frac{i}{2}\partial\bar{\partial}\log(1 - \|z\|^2)$ and the standard complex structure.

Of course, the main example we are interested in is the complex projective space:

Example 7.4 (Fubini-Study metric)**.** *On the complex projective space $\mathbb{CP}^n$, take canonical coordinates $[z_0 : z_1 : \cdots : z_n]$ and consider the usual trivialising charts $U_i = \{z_i \neq 0\} \cong \mathbb{C}^n$. On each U_i, we consider the forms*

$$\omega_i = \frac{i}{2\pi}\partial\overline{\partial}\log\left(\sum_{k=1}^{n}|w_k|^2 + 1\right) ,$$

where w_j are the standard coordinates of $\mathbb{C}^n \cong U_i$. We claim that the $\{\omega_i\}_i$ glue together to give a globally defined $(1,1)$-form on $\mathbb{CP}^n$, and that this form is positive with respect to the complex structure.

First, on $U_i \cap U_j$, we have that

$$\log\left(\sum_{k=0}^{n}\left|\frac{z_k}{z_i}\right|^2\right) = \log\left(\sum_{k=0}^{n}\left|\frac{z_j}{z_i}\right|^2\left|\frac{z_k}{z_j}\right|^2\right) = 2\big(\log|z_j| - \log|z_i|\big) + \log\left(\sum_{k=0}^{n}\left(\frac{z_k}{z_j}\right)^2\right) .$$

Thus, it suffices to show that $\partial\overline{\partial}\log|z|^2$ is zero. Indeed, one has

$$\partial\overline{\partial}\log|z|^2 = \partial\left(\frac{1}{z\overline{z}}\overline{\partial}(z\overline{z})\right) = \partial\left(\frac{d\overline{z}}{\overline{z}}\right) = 0 .$$

To check that ω is positive, we can use the local expression in U_i:

$$\partial\overline{\partial}\log\left(\sum_{k=1}^{n}|w_k|^2 + 1\right) = \frac{(1+|w_i|^2)\delta^{ij} - \overline{w_i}w_j}{(1+\sum_j|w_j|^2)^2}dw_i \wedge d\overline{w_j} .$$

In these coordinates, we have that for $u \in \mathbb{C}^n$,

$$2\pi\omega_i(u, Ju) = |u|^2 + |w|^2|u|^2 - (u,w)(w,u) > 0$$

where positivity follows from Cauchy–Schwarz.

The Fubini-Study metric in $\mathbb{CP}^n$ can also be realised as the restriction of the Kähler metric $\frac{i}{2\pi}\partial\overline{\partial}\log||z||^2$ on $\mathbb{C}^{n+1}$ under the natural projection map $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$. Moreover, the Fubini–Study metric can be viewed as a natural or canonical metric on $\mathbb{CP}^n$, as discussed in the exercises.

As in the case of almost complex submanifolds of complex manifolds, we have

Proposition 7.5. *Any submanifold of a Kähler manifold is Kähler. In particular, all projective manifolds are Kähler.*

This already gives a strong restriction on which manifolds can be projective. In particular, Hopf and Iwasawa manifolds are not projective.

7.1 The Kähler Identity and the Akizuki-Nakano Lemma

Let us prove the first key result in Kähler geometry, the so-called (first) Kähler identity:

Theorem 7.6. *Let (X, g, ω) be a compact Kähler manifold, and denote by Λ the adjoint of the Lefschetz operators. Then:*

$$[\Lambda, \overline{\partial}] = -i\partial^* \ . \tag{14}$$

Note that by conjugating and taking adjoints, we get three more equivalent identities.

$$[\Lambda, \partial] = i\overline{\partial}^* \qquad \text{(conjugation)} \ , \tag{15}$$

$$[L, \overline{\partial}^*] = -i\partial \qquad \text{(adjoint)} \ , \tag{16}$$

$$[L, \partial^*] = i\overline{\partial} \qquad \text{(conjugation and adjoint)} \ . \tag{17}$$

The collection of these four identities is then referred to as the Kähler identities.

We give the same proof of the Kähler identities as Huybrechts [Huy05]. However, there is an alternative, more intellectually satisfactory proof of the Theorem, but it requires some familiarity with Riemannian geometry. We give a sketch of it.

Lemma 7.7 (Akizuki–Nakano)**.** *The Kähler identities hold in $\mathbb{C}^n$ with its standard structure.*

Proof. We prove that $[L, \partial^*] = i\overline{\partial}$, with the other identities being equivalent.

Take flat coordinates $z_1, \ldots, z_n$, so $\omega = i \sum dz_k \wedge d\overline{z_j}$. For an arbitrary form α, we have

$$\partial^*\alpha = -\sum_{k=1}^{n} \frac{\partial}{\partial z_k} \lrcorner \left(\frac{\partial \alpha}{\partial \overline{z}_k}\right) \ .$$

Thus, we get

$$
\begin{aligned}
[L,\partial^*](\alpha) &= -\sum_{k=1}^{n} \omega\wedge \frac{\partial}{\partial z_k}\lrcorner\left(\frac{\partial\alpha}{\partial \overline{z}_k}\right) - \frac{\partial}{\partial z_k}\lrcorner\left(\frac{\partial(\alpha\wedge\omega)}{\partial \overline{z}_k}\right)\\
&= -\sum_{k=1}^{n} \omega\wedge \frac{\partial}{\partial z_k}\lrcorner\left(\frac{\partial\alpha}{\partial \overline{z}_k}\right) - \frac{\partial}{\partial z_k}\lrcorner\left(\omega\wedge\frac{\partial\alpha}{\partial \overline{z}_k}\right)\\
&= \sum_{k=1}^{n}\left(\frac{\partial}{\partial z_k}\lrcorner\omega\right)\wedge\frac{\partial\alpha}{\partial\overline{z}_k} = i\sum_{k=1}^{n} d\overline{z_k}\wedge\frac{\partial\alpha}{\partial\overline{z}_k}\\
&= i\overline{\partial}\alpha\ .
\end{aligned}
$$

□

We have established the Kähler identities for $\mathbb{C}^n$. The reader familiar with Riemannian geometry techniques can now see that

Corollary 7.8. *The Kähler identities hold for any Kähler manifold.*

The idea of the proof is to use normal coordinates, so the Kähler metric coincides with the standard Kähler metric up to second order. But the Kähler identities are first-order identities, so they must hold on any Kähler manifold. (Cf. [Dem12, Thm. 6.4] for details).

Let us now give a complete proof of the Kähler identities using Weil's formula, Corollary 6.14, which avoids the analytical technicalities of the construction of normal coordinates, as in [Huy05].

Proof of Theorem 7.6. First, by taking the conjugate of Equation (14), we get the equivalent identity

$$[\Lambda,d] = [\Lambda,\overline{\partial}] + [\Lambda,\partial] = i(\overline{\partial}^* - \partial^*) = -i*(\overline{\partial}-\partial)* = *(J\circ d\circ J)*\ , \tag{18}$$

where J is the induced almost complex structure acting on forms $\mathcal{A}_X^{p,q}$. We will prove this version of the Kähler identities.

By the Lefschetz decomposition, Proposition 6.11, it suffices to prove it for forms $L^j\alpha$ with $\alpha\in\Omega_0^k(X)$ a primitive section. Again by the Lefschetz decomposition, we can write $d\alpha=\sum_{j\geq 0}L^j\alpha_j$ with $\alpha_j\in\Omega_0^{k+1-2j}(X)$. By acting by L^{n-k+1}, we see that $\alpha_j=0$ for $j\geq 2$.

Let us compute the left and right-hand sides of the Kähler identity. First, we get

$$
\begin{aligned}
[\Lambda,d](L^j\alpha) &= \Lambda dL^j\alpha - d(\Lambda L^j\alpha) = \Lambda L^j d\alpha - d[\Lambda,L^j]\alpha\\
&= [\Lambda,L^j](\alpha_0+L\alpha_1) + j(n-k+j-1)L^{j-1}d\alpha\\
&= -jL^{j-1}\alpha_0 - (k-n+j-1)L^j\alpha_1\ ,
\end{aligned}
$$

where we used Corollary 6.9 repeatedly. Similarly, using Weil's Formula Corollary 6.14 repeatedly, we have

$$\begin{aligned}
*(J\circ d\circ J)(*L^j\alpha) &= (-1)^{\binom{k}{2}}\frac{j!}{(n-k-j)!}*(J\circ d\circ J)\left(L^{n-k-j}\left(J(\alpha)\right)\right)\\
&= (-1)^{\binom{k}{2}+k}\frac{j!}{(n-k-j)!}*L^{n-k-j}Jd\alpha\\
&= (-1)^{\binom{k}{2}+k}\frac{j!}{(n-k-j)!}*\left(L^{n-k-j}J\alpha_0+L^{n-k-j+1}J\alpha_1\right)\\
&= -jL^{j-1}\alpha_0-(k-n+j-1)L^j\alpha_1\ . \qquad \square
\end{aligned}$$

The Kähler identity extends naturally to the case of holomorphic vector bundles:

Lemma 7.9 (Akizuki-Nakano)**.** *Let $(E,h)\to X$ be a holomorphic hermitian bundle, with ∇ its Chern connection. Then*

$$[\Lambda,\overline{\partial}_E] = -i\left(\nabla^{1,0}\right)^*\ , \tag{19}$$

where $\left(\nabla^{1,0}\right)^* = -\overline{*}\,(\nabla^*)^{1,0}\,\overline{*}$ is the adjoint to $\nabla^{1,0}$, with ∇^* the induced connection on E^*.

Proof. The idea of the proof is to show that $[\Lambda,\overline{\partial}_E]+i\left(\nabla^{1,0}\right)^*$ is a zeroth-order operator, and then check that it is, in fact, zero.

Since the identity is local, it suffices to prove it on a trivialising cover $\{U_i\}$ of (E,h). On each U_i, there exists $A^{0,1}\in\mathcal{A}^{0,1}_X(U_i,E)$ such that on the U_i, we get the identification

$$\nabla = d+A = d+\overline{A^{0,1}}+A^{0,1} \qquad \nabla^* = d-A \qquad \overline{\partial}_E = \overline{\partial}+A^{0,1}\ .$$

In particular, we have $\left(\nabla^{1,0}\right)^* = -\overline{*}\left(\partial-\overline{A^{0,1}}\right)\overline{*} = \partial^*+*A^{0,1}*$. Thus, locally, we have

$$\begin{aligned}
[\Lambda,\overline{\partial}_E]+i\left(\nabla^{1,0}\right)^* &= [\Lambda,\overline{\partial}+A^{0,1}]+i\left(\partial^*+*A^{0,1}*\right)\\
&= \left([\Lambda,\overline{\partial}]+i\partial^*\right)+\left([\Lambda,A^{0,1}]+i*A^{0,1}*\right)\ .
\end{aligned}$$

The first term vanishing is precisely the Kähler identity (14). The second term is a zeroth-order term, and its vanishing is a straightforward linear exercise. $\square$

7.2 Consequences of the Kähler identities

The Kähler-Nakano identities have some useful consequences. First and foremost, we have the following Laplacian comparisons:

Theorem 7.10 (Bochner-Kodaira-Nakano)**.** *Let $(E,h)\to X$ be a holomorphic hermitian bundle, with Chern connection $\nabla_E = \partial_E+\overline{\partial}_E : \mathcal{A}^{p,q}\to\mathcal{A}^{p+1,q+1}$. Consider the Laplace-like operators*

$$\Delta_E = \nabla_E^*\nabla_E+\nabla_E\nabla_E^* \qquad \Delta_{\overline{\partial}_E} = \overline{\partial}_E^*\overline{\partial}_E+\overline{\partial}_E\overline{\partial}_E^* \qquad \Delta_{\partial_E} = \partial_E^*\partial_E+\partial_E\partial_E^*\ .$$

They satisfy

(i) $\Delta_{\overline{\partial}_E} = \overline{\Delta_{\partial_E}}$,

(ii) $[L,\Delta_{\overline{\partial}_E}] = -iF_{\nabla_E}$ *and* $[L,\Delta_{\partial_E}] = iF_{\nabla_E}$,

(iii) $\Delta_E = \Delta_{\partial_E} + \Delta_{\overline{\partial}_E}$, *and*

(iv) $\Delta_{\overline{\partial}_E} - \Delta_{\partial_E} = [iF_{\nabla_E} \wedge \cdot\,, \Lambda]$.

Proof. The first claim is straightforward. The two statements in (ii) are equivalent using (i). Let us prove $[L, \Delta_{\overline{\partial}_E}] = -iF_\nabla$:

$$[L, \Delta_{\overline{\partial}_E}] = [L, \overline{\partial}_E^*]\overline{\partial}_E + \overline{\partial}_E[L, \overline{\partial}_E^*] = -i\left(\partial_E\overline{\partial}_E + \overline{\partial}_E\partial_E\right) = -iF_\nabla ,$$

where we used

$$[L, \overline{\partial}_E^*] = [\Lambda, \overline{\partial}_E]^* = -i\,(\partial_E^*)^* = -i\partial_E ,$$

and $F_\nabla = (\partial_E + \overline{\partial}_E)^2 = \partial_E\overline{\partial}_E + \overline{\partial}_E\partial_E$. Let us prove that $\Delta_E = \Delta_{\partial_E} + \Delta_{\overline{\partial}_E}$. Indeed, we have

$$\begin{aligned}\Delta_E &= (\partial_E + \overline{\partial}_E)(\partial_E^* + \overline{\partial}_E^*) + (\partial_E^* + \overline{\partial}_E^*)(\partial_E + \overline{\partial}_E) \\ &= (\partial_E\partial_E^* + \partial_E^*\partial_E) + \left(\overline{\partial}_E\overline{\partial}_E^* + \overline{\partial}_E^*\overline{\partial}_E\right) + \left(\partial_E\overline{\partial}_E^* + \overline{\partial}_E\partial_E^* + \partial_E^*\overline{\partial}_E + \overline{\partial}_E^*\partial_E\right) \\ &= \Delta_{\partial_E} + \Delta_{\overline{\partial}_E} + (\partial_E\overline{\partial}_E^* + \overline{\partial}_E^*\partial_E) + \overline{(\partial_E^*\overline{\partial}_E + \overline{\partial}_E\partial_E^*)} .\end{aligned}$$

Thus, the claim follows from noticing that

$$\partial_E\overline{\partial}_E^* + \overline{\partial}_E^*\partial_E = i\partial_E[\partial_E, \Lambda] + i[\Lambda, \partial_E]\partial_E = i(\partial_E\Lambda\partial_E - \partial_E\Lambda\partial_E) = 0 .$$

Finally, to show (iv), notice that by virtue of the Akizuki-Nakano Lemma, we have

$$\begin{aligned}\Delta_{\partial_E} &= \partial_E\partial_E^* + \partial_E^*\partial_E = i\partial_E[\Lambda, \overline{\partial}_E] + i[\Lambda, \overline{\partial}_E]\partial_E \\ &= i\big(\partial_E\Lambda\overline{\partial}_E - \partial_E\overline{\partial}_E\Lambda + \Lambda\overline{\partial}_E\partial_E - \overline{\partial}_E\Lambda\partial_E\big) .\end{aligned}$$

We get a similar expression for Δ_{∂_E} by conjugation. Thus, we have

$$\begin{aligned}\Delta_{\overline{\partial}_E} - \Delta_{\partial_E} &= i\big(\overline{\partial}_E\partial_E\Lambda - \Lambda\partial_E\overline{\partial}_E\big) - i\big(\Lambda\overline{\partial}_E\partial_E - \partial_E\overline{\partial}_E\Lambda\big) \\ &= i\left(\overline{\partial}_E\partial_E + \partial_E\overline{\partial}_E\right)\Lambda - i\Lambda\left(\overline{\partial}_E\partial_E + \partial_E\overline{\partial}_E\right) \\ &= [iF_\nabla \wedge \cdot\,, \Lambda] .\end{aligned}$$

□

Of course, these formulas simplify significantly if the bundle E is flat. In particular, one gets

Corollary 7.11. *Let (X, g, ω) be a Kähler manifold. Then*

$$\Delta_{\overline{\partial}} = \Delta_\partial = \frac{1}{2}\Delta .$$

From the relation between the different Laplacians, we have

Corollary 7.12.

(i) For all $p, q \geq 0$, we have $\mathcal{H}^{p,q}_{\overline{\partial}}(X) \cong \mathcal{H}^{p,q}_{\partial}(X)$.

(ii) We have $\mathcal{H}^k_{dR}(X) \cong \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\overline{\partial}}(X)$.

Thus, there is a natural decomposition

$$H^k(X,\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X) \ .$$

One can show (cf. [Huy05, Cor 3.2.12]) that this decomposition is independent of the chosen Kähler metric.

By taking dimensions, we have the *refined Betti numbers*, $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}(X)$, which are usually arranged in the *Hodge diamond* as shown in Figure 1. The different dualities we have seen so far

$$\begin{array}{ccccccccc}
 & & & & h^{n,n} & & & & \\
 & & & h^{n,n-1} & & h^{n-1,n} & & & \\
 & & h^{n,n-2} & & h^{n-1,n-1} & & h^{n,-2} & & \\
 & & & & \vdots & & & & \\
h^{n,0} & & \cdots & & h^{p,q} & & \cdots & & h^{0,n} \\
 & & & & \vdots & & & & \\
 & & h^{2,0} & & h^{1,1} & & h^{0,2} & & \\
 & & & h^{1,0} & & h^{0,1} & & & \\
 & & & & h^{0,0} & & & &
\end{array}$$

Figure 1: Hodge diamond representation for a n-dimensional complex manifold.

translate into symmetries of the Hodge diamond:

- **Hodge duality:** $h^{p,q} = h^{n-q,n-p}$
- **Serre duality:** $h^{p,q} = h^{n-p,n-q}$
- **Conjugation:** $h^{p,q} = h^{q,p}$

Moreover, since Kähler manifolds are symplectic, we know that $h^{p,p} \geq 1$. Combining these with our knowledge of singular/de Rham cohomology, we immediately get the Hodge diamonds of $\mathbb{CP}^n$ and of a Riemann surface Σ^g. Another key result that follows from the Kähler identities is the famous $\partial\overline{\partial}$ Lemma:

Lemma 7.13 ($\partial\overline{\partial}$-lemma)**.** *Let X be a Kähler manifold, $\alpha \in \mathcal{A}^{p,q}_{X,closed}$. Then the following are equivalent:*

(i) α is exact,

(ii) α is ∂-exact,

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & & & & \vdots & & & & \\
0 & \cdots & & 0 & 1 & 0 & & \cdots & 0 \\
 & & & & \vdots & & & & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

Figure 2: Hodge diamond of $\mathbb{CP}^n$.

$$
\begin{array}{ccc}
 & 1 & \\
g & & g \\
 & 1 &
\end{array}
$$

Figure 3: Hodge diamond of a Riemann surface of genus g, Σ^g.

(iii) α *is* $\overline{\partial}$*-exact,*

(iv) α *is* $\overline{\partial}\partial$*-exact,*

(v) α *is orthogonal to* $\mathcal{H}^{p,q}_{\overline{\partial}}(X)$

Proof. First, $(i)-(iv)$ directly implies (v), and (iv) implies $(i)-(iii)$, so it suffices to show how (v) implies (iv).

By the Hodge $\overline{\partial}$-decomposition, there exists $\beta \in \mathcal{A}^{p,q-1}$ such that $\overline{\partial}\beta = \alpha$. Using the Hodge ∂-decomposition, we can find forms γ, γ' and γ'' such that $\beta = \partial\gamma + \partial^*\gamma + \gamma''$, with $\gamma \in \mathcal{H}^{p,q-1}_{\partial}$. Thus,

$$\alpha = \overline{\partial}\partial\gamma + \overline{\partial}\partial^*\gamma' \ .$$

Since α is closed, we have

$$0 = \langle \partial\alpha, \overline{\partial}\gamma' \rangle = \langle \partial\overline{\partial}\partial^*\gamma', \overline{\partial}\gamma' \rangle = -||\overline{\partial}\partial^*\gamma'||^2 \ ,$$

where we used the identity $\overline{\partial}\partial^* = -\partial^*\overline{\partial}$, which we saw in the previous proof, follows from the Kähler identities. So $\alpha = \overline{\partial}\partial\gamma$, as needed. $\square$

A direct corollary of the $\partial\overline{\partial}$-Lemma is

Corollary 7.14. *On a Kähler manifold, the natural maps $H^{p,q}_{BC} \to H^{p,q}_{\overline{\partial}}$ and $H^{p,q}_{\overline{\partial}} \to H^{p,q}_{A}$ are isomorphisms.*

The Kähler identity has one more important consequence:

Theorem 7.15 (Hard Lefschetz theorem)**.** *The Laplacian operator commutes with the Lefschetz operators. Thus, it induces an $\mathfrak{sl}(2,\mathbb{R})$ on $\mathcal{H}^*_{dR}$, which is compatible with the Lefschetz decomposition*

$$\mathcal{H}^k(X) \cong \bigoplus_{i\geq 0} L^i \mathcal{H}^{k-2i}(X)_P \ ,$$

where $\mathcal{H}^{k-2i}_0(X) = \mathcal{H}^{k-2i}(X) \cap P^k$ is the space of primitive harmonic forms.

Proof. By the Kähler identities, we have

$$\left[\Delta_{\overline{\partial}}, L\right] = \left[\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}, L\right] = \overline{\partial}[\overline{\partial}^*, L] + [\overline{\partial}^*, L]\overline{\partial} = i\left(\overline{\partial}\partial + \partial\overline{\partial}\right) = 0 \ .$$

By taking conjugates, we get $\left[\Delta_{\overline{\partial}}, \Lambda\right] = 0$, so the claim follows. ☐

In combination with Poincaré duality, the Hard Lefschetz theorem implies the following

Lemma 7.16. *Let X be an n-dimensional complex manifold and $\iota : Y \to X$ a complex hypersurface such that $[Y]$ is a Kähler class. Then the induced maps in cohomology*

$$\iota^* : H^k(X, \mathbb{C}) \to H^k(Y, C)$$

are injective for all $k \leq n-1$.

This is "half" of what is usually referred to as the Lefschetz Hyperplane Theorem, which we will prove in the next section.

We saw in Theorem 6.35 that the image of $c_1^{\mathbb{R}} : \mathrm{Pic}(X) \to H^2(X,\mathbb{R})$ is contained in $H^{1,1}(X,\mathbb{R})$. In the case of a Kähler manifold, we moreover have

Theorem 7.17 (Lefschetz (1,1)-theorem)**.** *Let X be a Kähler manifold, and consider the lattice*

$$H^{1,1}(X,\mathbb{Z}) \coloneqq \mathrm{Im}\left(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})\right) \cap H^{1,1}$$

Then, the image of the first Chern class $c_1 : \mathrm{Pic}(X) \to H^2(X,\mathbb{Z})$ contains the lattice $H^{1,1}(X,\mathbb{Z})$.

Proof. Let $\alpha \in \mathrm{Im}\left(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})\right)$. In particular, α is real, so have the bidegree decomposition $\alpha = \overline{\alpha^{0,2}} + \alpha^{1,1} + \alpha^{0,2}$. Thus, α is of type $(1,1)$ if and only if $\alpha^{0,2}$.

The long exact exponential sequence

$$H^1(X,\mathcal{O}_X) \to \mathrm{Pic}(X) \to H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X)$$

gives the desired equivalence. ☐

Note that a complex manifold can admit multiple Kähler metrics. Indeed, we define

Definition 7.18. Let (X, J, ω) a Kähler manifold. The *Kähler cone* is

$$\mathcal{K}_X \subseteq H^{1,1} \cap H^2(X, \mathbb{R}) \ ,$$

the subspace of all Kähler classes associated to a Kähler metric on (X, J).

Although we will not study the Kähler cone in detail, let us at least prove

Proposition 7.19. *The Kähler cone $\mathcal{K}_X$ is an open convex cone.*

Proof. Note that for ω_1 and ω_2 Kähler forms, $\lambda\omega_1$ and $\mu\omega_1 + (1-\mu)\omega_2$ are again Kähler forms. Now, positivity is an open condition, so the Kähler cone is open. □

We conclude this section by giving a proof of the signature theorem.

First, let us recall the definition of the signature. On a manifold M of (real) dimension $4m$, the Hodge star in middle degree is a symmetric endomorphism that squares to the identity. Thus, the space $\bigwedge^{2k} T^*M$ and $\mathcal{H}^{2m}(M)$ decomposes into 1 and (-1)-eigenspaces of the Hodge star. Thus, we have a refinement of the middle Betti number $b^{2m}(M) \coloneqq b_+(M) + b_-(M)$.

Definition 7.20. Let M be a manifold of (real) dimension $4m$. The *signature* of M is defined as

$$\operatorname{sign}(M) \coloneqq b_+(M) - b_-(M) \ .$$

It is not hard to show that $\operatorname{sign}(M)$ is in fact a topological invariant of M, up to orientation.

In the case of Kähler, the decomposition $b^{2m}(M) \coloneqq b_+(M) + b_-(M)$ is compatible with the Hodge decomposition $h^{p,q}(M)$. In fact, more is true:

Theorem 7.21 (Signature theorem)**.** *Let M be a Kähler manifold, with* $\dim_{\mathbb{C}}(M) = 2m$*. Then*

$$\operatorname{sign}(X) = \sum_{p,q=0}^{2m} (-1)^q h^{p,q}(X) \ .$$

Proof. Let $\mathcal{H}^{2m}(X, \mathbb{R}) = \mathcal{H}_+ \oplus \mathcal{H}_-$ the splitting by the Hodge star. Note that one may characterise

$$\mathcal{H}_+ \coloneqq \left\{ \alpha \in \mathcal{H}^{2m} \;\middle|\; \int_M \alpha \wedge \alpha \geq 0 \right\} \ ,$$

and similarly for $\mathcal{H}_-$. Integrating the Hodge-Riemann bilinear map form Proposition 6.16 implies that, for $\alpha \in \mathcal{H}_0^{p,q}(M)$, we have

$$\int_M \alpha \wedge \overline{\alpha} \wedge \omega^{2m-(p+q)} = i^{q-p} (-1)^{\binom{p+q}{2}} [2m - (p+q)]! \ ||\alpha||^2 \ .$$

The idea is to combine this with the Hard Lefschetz theorem, Theorem 7.15:

$$\mathcal{H}^{2m}(X,\mathbb{R}) = \bigoplus_{p+q=2m} \mathcal{H}^{p,q}(X)_{\mathbb{R}} = \mathcal{H}^{m,m}(X)_{\mathbb{R}} \oplus \bigoplus_{\substack{p+q=2m \\ p>q}} (\mathcal{H}^{p,q}(X) \oplus \mathcal{H}^{q,p}(X))_{\mathbb{R}}$$
$$= \bigoplus_{j\geq 0} L^j \mathcal{H}_0^{m-j,m-j}(X)_{\mathbb{R}} \oplus \bigoplus_{\substack{p+q=2m \\ p>q}} \bigoplus_{j\geq 0} L^j \left(\mathcal{H}_0^{p-j,q-j}(X) \oplus \mathcal{H}_0^{q-j,p-j}(X)\right)_{\mathbb{R}} ,$$

where each summand now has a definite sign under the intersection pairing by Proposition 6.16. Let us compute it. For $\alpha \in \mathcal{H}_0^{m-j,m-j}(X)$, we have $\int_M L^j\alpha \wedge L^j\alpha = \int_M \alpha \wedge \overline{\alpha} \wedge \omega^j$, and the Hodge-Riemann relations imply this has sign $(-1)^{m-j}$. Similarly, one checks that the pairing on $L^j \left(\mathcal{H}_0^{p-j,q-j}(X) \oplus \mathcal{H}_0^{q-j,p-j}(X)\right)$ has sign

$$i^{p-q}(-1)^{\binom{p+q-2j}{2}} = (-1)^{m+q-j} ,$$

since $p+q=2m$. Now, taking dimensions, we have

$$\begin{aligned}
\operatorname{sign}(X) &= \sum_{j\geq 0}(-1)^{m-j} h_0^{m-j,m-j} + 2\sum_{j\geq 0}(-1)^{m-j} \sum_{\substack{p+q=2m \\ p>q}} (-1)^q h_0^{p-j,q-j} \\
&= \sum_{p+q=2m} (-1)^q \sum_{j\geq 0}(-1)^{m-j} h_0^{p-j,q-j} \\
&= \sum_{p+q=2m} (-1)^q \left(h^{p,q} + 2\sum_{j>0}(-1)^j h^{p-j,q-j} \right) \\
&= \sum_{p,\, q=0}^{2m} (-1)^q h^{p,q} ,
\end{aligned}$$

where to go from the second to the third line we used the fact that $h^{p,q} = \sum_{j\geq 0} h_0^{p-j,q-j}$, and the fourth line follows by using the symmetries of the Hodge numbers and linear algebra. □

In the case of complex surfaces, this yields:

Corollary 7.22. *Let S be a compact complex surface. Then*

$$b_+ = 2h^{2,0} + 1 \qquad\qquad b_- = h_{1,1} - 1$$

Proof. From the Hodge decomposition and the signature theorem, we have

$$b_2 = b_+ + b_- = 2h^{2,0} + h^{1,1} \qquad \operatorname{sign}(X) = b_+ - b_- = 2 - h^{1,1} + 2h^{2,0} .$$ □

7.3 The Hirzebruch–Riemann–Roch Theorem

Let us now briefly discuss the Hirzebruch–Riemann–Roch theorem. The result, in our case, is a consequence of the more general Atiyah–Singer Index Theorem for Fredholm operators. This result stands as a cornerstone of modern differential geometry, providing a bridge between analytic and topological data. However, its proof is beyond the scope of these notes. Let us introduce the main character of the discussion:

Definition 7.23. Let $E \to X$ a holomorphic vector bundle. Its *holomorphic Euler characteristic* is defined as

$$\chi(X, E) = \sum_{q \geq 0} (-1)^q \dim H^q(X, E) \ .$$

Let us give an analytic interpretation of the holomorphic Euler characteristic in the case where X is Kähler. Consider the "rolled-up" operator

$$D = \overline{\partial}_E + \overline{\partial}_E^* : \bigoplus_{q \ even} \mathcal{A}_X^{0,q}(E) \to \bigoplus_{q \ odd} \mathcal{A}_X^{0,q}(E) \ .$$

Then, by the Hodge decomposition, the kernel and cokernel of D are finite, given by

$$\ker(D) = \bigoplus_{q \ even} \mathcal{H}^{0,q}(X, E) \qquad \qquad \mathrm{coker}(D) = \bigoplus_{q \ odd} \mathcal{H}^{0,q}(X, E) \ .$$

Therefore, its index is precisely the holomorphic Euler characteristic:

$$\mathrm{ind}(D) = \dim \ker(D) - \dim \mathrm{coker}(D) = \sum_{q \geq 0} (-1)^q \dim \mathcal{H}^{0,q}(X, E) = \chi(X, E) \ .$$

The Atiyah–Hitchin index theorem states that for the operator D (as an elliptic differential operator), its analytic index is equal to its topological index, a quantity that depends solely on the underlying topology of E (and X). In our case, we have

Theorem 7.24 (Hirzebruch–Riemann–Roch)**.** *The holomorphic Euler characteristic of a holomorphic vector bundle E satisfies*

$$\chi(X, E) = \int_X \mathrm{ch}(E)\,\mathrm{Td}(X) \ ,$$

where ch *and* Td *are the Chern character and Todd class, defined in Section 6.5.*

Note that $\mathrm{ch(E)}\,\mathrm{Td}(X)$ is a cohomology class of mixed degree, and integrating over the manifold X corresponds to pairing with the fundamental class of the manifold, which only detects the top degree part of the (mixed) cohomology class,

$$[\mathrm{ch(E)}\,\mathrm{Td}(X)]_{2n} = \sum_{k=0}^{n} \mathrm{ch}_i(E) \wedge \mathrm{Td}_{k-i}(X) \ .$$

We will make use of the Hirzebruch–Riemann–Roch theorem in Section 10.2. For now, we give a proof of the Hirzebruch signature theorem for complex manifolds:

Theorem 7.25 (Hirzebruch)**.** *Let X^{2n} be a compact Kähler manifold of complex dimension $2n$. Its signature $\sigma(X) = b_+ - b_-$ is a topological invariant of X, and satisfies*

$$\sigma(X) = \int_X L(TX) \ ,$$

where L is the L-genus:

$$L = 1 + \frac{1}{3}\big(c_1^2 - 2c_2\big) + \frac{11c_2^2 - 14c_1c_3 + 14c_4 + 4c_2c_1^2 - c_1^4}{45} + \dots$$

Proof. By the signature Theorem 7.21, we have

$$\sigma(X) = \sum_{q=0}^{2n} (-1)^q h^{p,q} = \sum_{p=0}^{2n} \chi(X, \Omega_X^p) \ .$$

The Hirzebruch–Riemann–Roch Theorem 7.24 implies

$$\sigma(X) = \sum_{p=0}^{2n} \int_X \mathrm{ch}(\Omega_X^p) \, \mathrm{Td}(TX) \ .$$

Thus, $\sigma(X)$ is a topological invariant of X. The formula for the L-genus can be computed by the expression $L(TX) = \mathrm{ch}(\Omega_X^p) \, \mathrm{Td}(TX)$ using the Chern roots of TX. □

8 Positivity and the Kodaira Embedding Theorem

Throughout this section, the Kähler manifold will be assumed to be closed, that is, compact and without boundary. We now introduce the last notion that will play a key role in the Kodaira embedding theorem. First, recall that a real $(1,1)$-form α is called positive if $\alpha(v, Jv) > 0$. Equivalently, taking the complexification, we have $-i\alpha(v, \overline{v}) > 0$ for all $v \in T^{1,0}X$, $v \neq 0$.

Definition 8.1. Let $(E, h) \to X$ be a hermitian vector bundle and ∇ its Chern connection, with curvature $F_\nabla \in A^{1,1}(\mathrm{End}(E))$. We say (E, h) is positive if its curvature is a positive form, that is,

$$h(\Omega_\nabla(s), s)(v, \overline{v}) > 0 \quad \text{for all } v \in T^{1,0}X, s \in \Gamma(U, E), s \neq 0.$$

Similarly, we define semi-positivity and (semi-)negativity. In the special case of a line bundle, we have

Lemma 8.2. *If E is a line bundle and $\nabla = \nabla^{ch}$ is the Chern connection, then Ω_∇ is positive if and only if $i\Omega_\nabla$, as a 2-form, is positive.*

Proof. We have

$$h(\Omega_\nabla(s), s) = h(s, s) \cdot \Omega_\nabla(v, \bar{v}) = (-i)(i\Omega_\nabla(v, \bar{v})) \cdot h(s, s),$$

so $h(\Omega_\nabla(s), s) > 0$ if $i\Omega_\nabla(v, \bar{v}) > 0$. Alternatively, one can use the identification $\mathfrak{u}(1) \cong i\mathbb{R}$. □

Note that the existence of a positive line bundle $L \to X$ implies that X is Kähler, with Kähler form $\omega = \frac{i}{2\pi}\Omega$. In particular, it follows from Exercise 4 in Sheet 7 that

Proposition 8.3. *The line bundle $\mathcal{O}_{\mathbb{CP}^n}(1)$ is a positive line bundle, and the induced Kähler metric is the Fubini–Study metric.*

It is important to remark that positivity of a line bundle is a purely topological property of L as shown by the next lemma:

Lemma 8.4. *Let $L \to X$ be a holomorphic line bundle on a Kähler manifold X. For any real closed $(1,1)$-form α with $[\alpha] = c_1(L)$, there exists a hermitian metric on L such that the curvature of its Chern connection is $\frac{2\pi}{i}\alpha$.*

Proof. Consider (L, h) a holomorphic line bundle equipped with a hermitian metric h. From the definition of the Chern connection, we know that its curvature is given by $F_\nabla = \overline{\partial}\partial \log(h)$.

Any other metric is given by $h' = e^f h$ for $f \in \mathcal{A}^0_{X,\mathbb{R}}$, so the curvature of the two Chern connections is related by $F_{\nabla'} = \overline{\partial}\partial \log(e^f h) = \overline{\partial}\partial f + F_\nabla$.

Thus, we would want to solve the equation $\alpha = \overline{\partial}\partial f + F_\nabla$ for f given that $\frac{i}{2\pi}[\alpha] = c_1(L)$.

But, by Theorem 6.35, we know that $[F_\nabla - \alpha] = 0$, so the difference is d-exact, and by the $\partial\overline{\partial}$-lemma 7.13, it is $\partial\overline{\partial}$-exact, as needed. □

Note that the argument above also shows that $c_1(L)$ is a well-defined element in $H^{1,1}_{BC}(X)$ for any complex manifold, without the Kähler assumption. More generally, for $E \to X$ a holomorphic vector bundle, $[c_k(E, \nabla)]$ is a well-defined element in $H^{k,k}_{BC}(X)$.

Let us now study how the positivity condition restricts to subbundles. First, we need

Definition 8.5. Let $E \to X$ be a bundle with connection ∇ and $F \subset E$ a complex subbundle. The *fundamental form* of F in E is a section of $\mathrm{Hom}(F, E/F)$, defined as:

$$\begin{aligned} b : \mathcal{A}^0_X(F) &\to \mathcal{A}^0_X(E/F) \\ s &\mapsto \pi_{E/F}(\nabla s) \end{aligned}$$

As in the usual case, one readily verifies that this is a well-defined section of $\mathrm{Hom}(F, E/F)$.

If $(E, h) = (E_1, h_1) \oplus (E_2, h_2)$ with metric compatible connections $\nabla = \nabla_1 \oplus \nabla_2$, then simply $b_i(s) = \nabla(s) - \nabla_i(s)$, and from the metric compatibility, one has the relation

$$h_1(s, b_2(t)) + h_2(b_1(s), t) = 0 \tag{20}$$

for $s \in \mathcal{A}^0_X(E_1)$ and $t \in \mathcal{A}^0_X(E_2)$. Using this, we can prove

Theorem 8.6. *Consider*

$$0 \to E_1 \to \mathcal{O}^r_X \to E_2 \to 0$$

a short exact sequence of holomorphic hermitian bundles, with the standard metric $\mathcal{O}^r_X$, and the induced metrics on E_i. Then (E_1, h_1) is semi-negative and (E_2, h_2) is semi-positive.

Proof. Choose a (smooth) splitting $\mathcal{O}^r_X \cong E_1 \oplus E_2$, so the Chern connection and its curvature become

$$\nabla = \begin{pmatrix} \nabla_1 & b_2 \\ b_1 & \nabla_2 \end{pmatrix} \qquad F_\nabla = \begin{pmatrix} \nabla_1 & b_2 \\ b_1 & \nabla_2 \end{pmatrix}^2 = \begin{pmatrix} F_{\nabla_1} + b_2 b_1 & \nabla_1 b_2 + b_2 \nabla_2 \\ b_1 \nabla_1 + \nabla_2 b_1 & F_{\nabla_2} + b_1 b_2 \end{pmatrix}.$$

But since $\mathcal{O}^r_X$ is trivial, so $\nabla = d$ and $F_\nabla = 0$, so $F_{\nabla_1} = -b_2 b_1$. Using the metric compatibility for the fundamental forms, Equation (20), we have

$$h_1\big(F_{\nabla_1}(s), s\big) = -h_1\big(b_2 b_1(s), s\big) = -h_2\big(b_1(s), b_1(s)\big).$$

Now, the claim follows since $b_1(s)$ is of type $(1,0)$, as the Chern connections satisfies $\nabla^{0,1} = \overline{\partial}_E$.

Explicitly, for the standard basis $\{e_1, \ldots, e_r\}$ of $H^0(X, \mathcal{O}_X^r)$, we can write $b_1(s) = \sum_i \alpha_i e_i$ for $\alpha_i \in \mathcal{A}_X^{1,0}$, and so $h_2\big(b_1(s), b_1(s)\big) = h\big(b_1(s), b_1(s)\big) = \sum_i \alpha_i \wedge \overline{\alpha_i}$.

The claim for E_2 follows by taking the dual sequence

$$0 \to E_2^* \to \mathcal{O}_X^r \to E_1^* \to 0\ . \qquad \square$$

We now have one more consequence of the 7.10

Lemma 8.7. *Let $(E,h) \to X$ be a holomorphic hermitian, equipped with its Chern connection ∇, with curvature F_∇. For any $\alpha \in \mathcal{H}^{p,q}_{\overline{\partial}_E}$, we have $\big([\Lambda, iF_\nabla](\alpha), \alpha\big)_{L^2} \geq 0$.*

Proof. Recall the Bochner-type comparison formula $\Delta_{\overline{\partial}_E} - \Delta_{\partial_E} = [iF_\nabla \wedge \cdot\ , \Lambda]$. Thus, for $\alpha \in \mathcal{H}^{p,q}_{\overline{\partial}_E}(X, E)$, we have

$$\big([iF_\nabla, \Lambda]\alpha, \alpha\big) = (-\Delta_{\partial_E}\alpha, \alpha) = -||\partial_E \alpha||^2 \leq 0\ . \qquad \square$$

Using this result together with the positivity condition, we have the celebrated Serre vanishing theorem:

Theorem 8.8 (Serre vanishing theorem)**.** *Let X $E \to X$ be a holomorphic vector bundle and $L \to X$ a positive line bundle. There exists a constant $m_0 \geq 0$ such that*

$$H^q(X, E \otimes L^m) = 0$$

for all $m \geq m_0$ and $q > 0$.

Proof. Pick hermitian metrics h_E and h_L on E and L respectively. We denote their Chern connections by ∇_E and ∇_L, and their curvatures F_{∇_E} and F_{∇_L}. By the positivity assumption on L, we know that $\omega = \frac{i}{2\pi} F_{\nabla_L}$ defines a Kähler metric on X.

Take $F = E \otimes K_X^* \otimes L^m$. Then, for $\alpha \in \mathcal{H}^{n,q}_{\overline{\partial}_F}(X, F)$, Lemma 8.7 implies

$$\begin{aligned}
0 \leq & \frac{i}{2\pi}\big([\Lambda, F_{\nabla_F}](\alpha), \alpha\big) && (21)\\
= & \frac{i}{2\pi}\big([\Lambda, F_{\nabla_{E\otimes K_X^*}}](\alpha), \alpha\big) + \frac{i}{2\pi}\big([\Lambda, F_{\nabla_{L^m}}](\alpha), \alpha\big) && (22)\\
= & \frac{i}{2\pi}\big([\Lambda, F_{\nabla_{E\otimes K_X^*}}](\alpha), \alpha\big) + m\big([\Lambda, L](\alpha), \alpha\big) && (23)\\
\leq & \frac{1}{2\pi} |||[\Lambda, F_{\nabla_{E\otimes K_X^*}}]|||\ ||\alpha||^2 + m\big(n - (n+q)\big)||\alpha||^2 && (24)\\
= & \left(|||[\Lambda, F_{\nabla_{E\otimes K_X^*}}]||| - 2\pi m q \right) \frac{||\alpha||^2}{2\pi}\ . && (25)
\end{aligned}$$

Thus, if $|||[\Lambda, F_{\nabla_{E\otimes K_X^*}}]||| < 2\pi m q$, we have

$$0 = \mathcal{H}^{n,q}(X, E \otimes K_X^* \otimes L^m) = \mathcal{H}^{0,q}(X, E \otimes L^m) = H^q(X, E \otimes L^m)\ . \qquad \square$$

If one takes $E = \underline{\mathbb{C}}$, so $F_{\nabla_E} = 0$, the proof above yields

Theorem 8.9 (Kodaira vanishing)**.** *Let $L \to X$ be a positive line bundle on a compact complex manifold X. Then:*

$$H^q(X, \Omega^p_X \otimes L) = 0 \quad \textit{for } p + q > \dim_{\mathbb{C}} X.$$

Proof. Consider the a Kähler metric on X given by $\omega = \frac{i}{2\pi} F_{\nabla_L}$. As before, for $\alpha \in \mathcal{H}^{p,q}_{\overline{\partial}_L}(X, L)$, we have

$$0 \le \frac{i}{2\pi} \left([\Lambda, F_{\nabla_L}] (\alpha), \alpha \right) = \frac{1}{2\pi} \big([\Lambda, L](\alpha), \alpha \big) = (n - (p+q)) ||\alpha||^2 \,. \qquad \square$$

Similarly, Serre duality implies that, for a negative line bundle L, one has $H^q(X, \Omega^p_X \otimes L) = 0$ for $p + q < \dim_{\mathbb{C}} X$, since

$$H^q(X, \Omega^p_X \otimes L) \cong H^{n-q}(X, \Omega^{n-p}_X \otimes K_X \otimes L^*)^*.$$

8.1 First consequences of positivity

Let us investigate some consequences of the Serre and Kodaira vanishing theorems. First, Kodaira vanishing allows us to prove the vanishing of the higher cohomology groups of $\mathcal{O}(m)$:

Proposition 8.10. *For $\mathcal{O}(m) \to \mathbb{CP}^n$, we have*

$$H^q(\mathbb{P}^n, \mathcal{O}(m)) = \begin{cases} \mathbb{C}[z_0, \dots, z_n]_m & \textit{if } q = 0,\ m \ge 0\,, \\ \mathbb{C}[z_0, \dots, z_n]_{(-n-1-m)} & \textit{if } q = n,\ m \le -n-1\,, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. First, note that $\mathcal{O}(m)$ is positive for $m > 0$ since $\mathcal{O}(1)$ is positive. Hence:

$$H^q(\mathbb{P}^n, \Omega^p \otimes \mathcal{O}(m)) = 0 \quad \text{for } m > 0,\ p + q > n\,,$$

by Kodaira vanishing. In particular, since $K_{\mathbb{CP}^n} \cong \mathcal{O}(-n-1)$, we have

$$H^q(\mathbb{P}^n, \mathcal{O}(m)) = 0 \quad \text{for } m > -n \ \text{ and } \ q > 0.$$

By Serre duality, we obtain the remaining terms. $\square$

Kodaira vanishing allows us to complete the proof of the Lefschetz Hyperplane Theorem:

Theorem 8.11 (Lefschetz Hyperplane Theorem)**.** *Let X be a compact Kähler manifold of dimension n, and $Y \subset X$ a smooth hypersurface such that $\mathcal{O}(Y)$ is positive. Then the restriction:*

$$H^k(X, \mathbb{C}) \to H^k(Y, \mathbb{C})$$

is an isomorphism for $k \le n-2$, and injective for $k = n-1$.

Proof. Use Hodge decomposition $H^k = \bigoplus H^{p,q}$. It suffices to show that

$$H^q(X, \Omega^p_X) \to H^q(Y, \Omega^p_Y)$$

is injective for $p + q = n - 1$, and bijective for $p + q \leq n - 2$.

We have exact sequences:

$$0 \to \mathcal{O}_X(-Y) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0 \qquad \text{and} \qquad 0 \to T_Y \to T_X|_Y \to N_{Y/X} \to 0,$$

and the dual of the latter

$$0 \to \mathcal{O}_Y(-Y) \to \Omega^1_X|_Y \to \Omega^1_Y \to 0.$$

More generally, we have:

$$0 \to \Omega^p_X(-Y) \to \Omega^p_X \to \Omega^p_X|_Y \to 0 \tag{26a}$$

$$0 \to \Omega^{p-1}_Y(-Y) \to \Omega^p_X|_Y \to \Omega^p_Y \to 0. \tag{26b}$$

The long exact sequence of (26a) is

$$\cdots \to H^q(X, \Omega^p_X(-Y)) \to H^q(X, \Omega^p_X) \to H^q(Y, \Omega^p_X|_Y) \to H^{q+1}(X, \Omega^p_X(-Y)) \to \ \ldots\ .$$

Since $\mathcal{O}(-Y)$ is a negative line bundle, $H^q(X, \Omega^p_X(-Y) = H^{n-q}(X, \Omega^{n-p}_X \otimes \mathcal{O}(Y))^* = 0$ for $q < n$ by Kodaira vanishing. Hence, $H^q(X, \Omega^p_X) \to H^q(Y, \Omega^p_X|_Y)$ is injective for $q < n$, and bijective for $q < n - 1$. Similarly, the long exact sequence of (26b) is

$$\cdots \to H^q(Y, \Omega^{p-1}_Y(-Y)) \to H^q(Y, \Omega^{p-1}_X|_Y) \to H^q(Y, \Omega^p_Y) \to H^{q+1}(Y, \Omega^{p-1}_Y(-Y)) \to \ldots$$

Again, Kodaira vanishing implies $H^q(Y, \Omega^{p-1}_X|_Y) \to H^q(Y, \Omega^p_Y)$ is injective for $q = n - 1$, and bijective for $q < n - 1$. □

Thus, the Hodge diamond of a hypersurface coincides with the Hodge diamond of the ambient manifold, away from the middle dimension. In particular, a projective has a Hodge diamond as shown in Figure 4:

We conclude this section by giving a proof that the divisor class group (cf. Definition 5.35 and the Picard group are isomorphic for projective manifolds. First, we need the following lemma, which is of interest in its own right:

Lemma 8.12. *Let $E \to X$ be a holomorphic vector bundle and L a positive line bundle. Then $H^0(X, E \otimes L^k) \neq 0$ for $k \gg 0$ large enough.*

Proof. The idea is to combine By Serre's Vanishing Theorem 8.8 with the Hirzebruch–Riemann–Roch Theorem 7.24.

Consider the holomorphic Euler characteristic of $E \otimes L^k$. By Serre's Vanishing Theorem 8.8, all higher cohomology groups vanish for k large enough, so $\chi(X, E \otimes L^k) = H^0(X, E \otimes L^k)$. Using the

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & & & & \vdots & & & & \\
h^{n-1,0} & & \cdots & & h^{p,n-1-p} & & \cdots & & h^{0,n-1} \\
 & & & & \vdots & & & & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

Figure 4: Hodge diamond of a hypersurface in $\mathbb{CP}^n$.

Hirzebruch–Riemann–Roch Theorem 7.24, it follows that $\chi(X, E \otimes L^k)$ is a polynomial in k:

$$
\begin{aligned}
H^0(X, E \otimes L^k) &= \int_X \mathrm{ch}(E \otimes L^k)\,\mathrm{Td(X)} = \int_X \mathrm{ch}(E) e^{kc_1(L)}\,\mathrm{Td(X)} \\
&= \int_X \left(\mathrm{rk}(E) + \sum_{j=1}^n c_j(E)\right)\left(\sum_{j=0}^n \frac{[kc_1(L)]^j}{j!}\right)\mathrm{Td(X)} \\
&= k^n\,\mathrm{rk}(E)\int_X \frac{c_1^n(L)}{n!} + k^{n-1}\int_X \big(c_1(E) + \mathrm{rk}(E)c_1(X)\big)\frac{c_1^{n-1}(L)}{(n-1)!} + O(k^{n-2})\ ,
\end{aligned}
$$

where $n = \dim(X)$. Since L is positive, $\langle c_1(L)^n, [X]\rangle > 0$ and therefore $\chi(X, E \otimes L^k) > 0$, as needed. □

We can now prove

Theorem 8.13. *For $X \subseteq \mathbb{CP}^n$ smooth complex submanifold, we have $Cl(X) \cong Pic(X)$.*

Proof. We need to show that every line bundle comes from a divisor. Let $L \in \mathrm{Pic}(X)$. By the previous lemma, there exists $0 \neq s \in H^0(X, L(k))$ for k large enough, and by the line bundle-divisor correspondence (cf.Proposition 5.36), we have $L(k) \cong \mathcal{O}\big((s)\big)$.

Take $t \in H^0(X, \mathcal{O}(1))$, and consider the divisor $D = (s) - (t^k)$. Then $\mathcal{O}(D) = L(k) \otimes \mathcal{O}(-k) \cong L$ as needed. □

As a direct corollary, we get the projective Lefschetz (1,1)-theorem

Corollary 8.14. *For any projective manifold X, there is an isomorphism*

$$H^{1,1}(X,\mathbb{Z}) \cong \langle \textit{divisors} \rangle \subset H^2(X,\mathbb{R})$$

under the Poincaré map $\eta : \mathrm{Div}(X) \to H^2(X,\mathbb{Z})$.

In layman's terms, this says that every integral cohomology class of type $(1,1)$ of a projective manifold X is the image of a divisor in X under the Poincaré map.

One can ask the corresponding for higher degree. Specifically, set $H^{p,p}(X,\mathbb{Z}) := H^{p,q}(X,\mathbb{C}) \cap \mathrm{Im}\big(H^{2k}(X,\mathbb{Z}) \to H^{2k}(X,\mathbb{C})\big)$. Then one can ask if $H^{p,p}(X,\mathbb{Z})$ is generated by complex submanifolds of codimension p. The answer in this case is no, as shown by Atiyah and Hirzebruch, and Kollár with different types of counterexamples.

However, in both cases, the counterexamples disappear if one considers $H^{p,p}(X,\mathbb{Q}) := H^{p,q}(X,\mathbb{C}) \cap \mathrm{Im}\big(H^{2k}(X,\mathbb{Q}) \to H^{2k}(X,\mathbb{C})\big)$. The corresponding statement is the famous Hodge conjecture, one the seven millennium Clay problems.

Conjecture 8.15. *The group $H^{p,p}(X,\mathbb{Q})$ is generated over the rationals by the cohomology classes of complex submanifolds of X.*

8.2 The Kodaira embedding theorem

We are now ready to state and prove the Kodaira embedding theorem.

Theorem 8.16 (Kodaira embedding theorem)**.** *Let X be a compact manifold, and $L \to X$ a positive holomorphic line bundle. Then the pluricanonical map of L^k, $\phi_{L^k} : X \to \mathbb{CP}^N$ is an embedding for some $k \gg 0$ large enough.*

The converse of the Kodaira embedding theorem is immediately true, by considering $L = \phi^*\big(\mathcal{O}(1)\big)$.

Let us recall the framework of pluricanonical maps. Given a line bundle $L \to X$ and $\{s_i\}_{i=1,\dots,k}$ a basis of $H^0(X,L)$, we have a holomorphic map

$$\phi_L : X \setminus B_L \to \mathbb{CP}^N ,$$

which is independent of the chosen basis, up to a biholomorphism of $\mathbb{CP}^N$. The map ϕ_L will be an embedding whenever the following three conditions are satisfied:

(I) $B_L = \emptyset$,

(II) φ_L is injective, and

(III) $d\varphi_L$ is injective.

The first step in the proof of the embedding theorem is to give a cohomological equivalent to these conditions. We have

Proposition 8.17. *The conditions (I) - (III) are equivalent to*

(I_H) $H^1\big(X,\mathcal{I}_x(L)\big) = 0$ *for all $x \in X$,* (II$_H$) $H^1\big(X, \mathcal{I}_{x,y}(L)\big) = 0$ *for all* $x, y \in X$, *and*

(III$_H$) $H^1\big(X, \mathcal{I}_x^2(L)\big) = 0$ *for all* $x \in X$.

where $\mathcal{I}_Z$ *is the ideal sheaf of* $Z \subseteq X$, *and* $\mathcal{I}_Z(L) \coloneqq \mathcal{I}_Z \otimes L$.

Proof. First, the condition that the base locus B_L is empty is equivalent to the restriction map $H^0(X, L) \to L_x$ being surjective for every point $x \in X$. Recall the short exact sequence of sheaves:

$$0 \to \mathcal{I}_x \to \mathcal{O}_X \to S_X(x) \to 0 \ ,$$

where $S_X(x)$ is the skyscraper sheaf supported at $x \in X$. Tensoring with L, we get the short exact sequence

$$0 \to \mathcal{I}_x(L) \to L \to L_x \to 0 \ .$$

Taking cohomology, we get the long exact sequence

$$0 \to H^0(X, \mathcal{I}_x(L)) \to H^0(X, L) \to L_x \to H^1(X, \mathcal{I}_x(L)) \to H^1(X, L) \to \dots \ .$$

Thus, $H^1(X, \mathcal{I}_x) = 0$ is a sufficient condition for the map $H^0(X, L) \to L_x$ to be surjective.

Similarly, ϕ_L is injective if and only if the map $H^0(X, L) \to L_x \oplus L_y$ is surjective. The claim follows by using the short exact sequence

$$0 \to \mathcal{I}_{x,y}(L) \to L \to L_x \oplus L_y \to 0 \ .$$

Finally, let us assume that ϕ_L is injective. For $x \in X$, take $s_0 \in H^0(X, L)$ such that $s_0(x) \neq 0$. We claim that the differential $d\phi_L$ is injective at $x \in X$ if the map

$$\begin{aligned} d_x : H^0(X, L \otimes \mathcal{I}_x) &\to L_x \otimes (T^*_{\mathbb{C}} X)^{1,0} \\ s &\mapsto [(\psi \circ s_0)^{-1} d(\psi \circ s)]_x \end{aligned}$$

being surjective, where ψ is a choice of local trivialisation. Note that

$$H^0(X, L \otimes \mathcal{I}_x) = \{\text{sections of } L \text{ vanishing at } x\} \ ,$$

and the map d_x is well-defined: for a different trivialisation $\psi' = \lambda\psi$, we have

$$\begin{aligned} [(\psi' \circ s_0)^{-1} d(\psi' \circ s)]_x &= [(\lambda\psi \circ s_0)^{-1} d(\lambda\psi \circ s)]_x = [\lambda^{-1}(\psi \circ s_0)^{-1} \left(\lambda d(\psi \circ s) + (\psi \circ s) d\lambda\right)]_x \\ &= [(\psi \circ s_0)^{-1} d(\psi \circ s)]_x \ , \end{aligned}$$

since $(\psi \circ s)_x = 0$ as $s \in L \otimes \mathcal{I}_x$. A different choice of s_0 gives an isomorphism on $\to L_x \otimes (T^*_{\mathbb{C}} X)^{1,0}$. Take a basis $\{s_1, \dots s_k\}$ of $H^0(X, L \otimes \mathcal{I}_x)$, and consider $t_i = \frac{s_i}{s_0}$. In a neighbourhood U of x, ϕ_L is given by

$$\begin{aligned} U &\to \mathbb{C}^k \\ y &\mapsto (t_1(y), \dots, t_k(y)) \ . \end{aligned}$$

The condition that $d\phi_L$ is injective is equivalent to the 1-forms $dt_1, \ldots, dt_k$ spanning $T_x^{1,0}X$, i.e. the map d_x being surjective.

Finally, we claim that there is a short exact sequence of sheaves

$$0 \to \mathcal{I}_x^2 \to \mathcal{I}_x \xrightarrow{d_x} (T^*_{\mathbb{C}}X)^{1,0} \ . \tag{27}$$

The map d_x is well-defined: for $f, g \in \mathcal{I}_x$, we have

$$d(fg)_x = df_x g_x + g_x df_x = 0 \ .$$

It is also bijective: locally, we can take coordinates $\{z_1, \ldots z_n\}$, so $x = (0, \ldots, 0)$, $\mathcal{I}_0 = \langle z_1, \ldots, z_n \rangle$, and

$$\mathcal{I}_0/\mathcal{I}_0^2 \cong \mathbb{C}[dz_1, \ldots, dz_n] = (T^*\mathbb{C}^n)^{1,0} \ .$$

The long exact sequence of cohomology of the short exact sequence (27) twisted by L, gives the condition $H^1(X, \mathcal{I}_x^2(L)) = 0$ □

We have a characterisation of the pluricanonical ϕ_L being an embedding in terms of the vanishing of certain cohomology groups. Thus, we need to show that they indeed vanish for L^k with k large enough, provided L is positive. Of course, the idea is to use Kodaira vanishing, Theorem 8.8.

In the case of Riemann surfaces, points are divisors, and $\mathcal{I}_x \cong \mathcal{O}(-x)$ by virtue of Lemma 5.38. Moreover, one has a commutative diagram

$$\begin{array}{ccc} \mathrm{Cl}(X) & \longrightarrow & \mathrm{Pic}(X) \\ & \searrow^{\deg} & \downarrow{\scriptstyle c_1} \\ & & H^2(X, \mathbb{Z}) \cong \mathbb{Z} \end{array}$$

where $\deg\left(\sum a_i[x_i]\right) \coloneqq \sum a_i$ is the *degree* map. In particular, a line bundle L is positive if its degree is positive. Using this, one can directly prove:

Theorem 8.18. *Let $L \to \Sigma$ be a line bundle on a Riemann surface Σ such that*

$$\deg(L) > \deg(K_\Sigma) + 2 \ .$$

Then its pluricanonical map ϕ_L is an embedding.

The proof is left as an exercise to the reader (cf. Exercise 56).

For higher dimensions, one faces the challenge that the ideal $\mathcal{I}_x$ is not locally free, so it cannot be identified directly with a line bundle, and we cannot use the Kodaira vanishing theorem on the nose. However, if we blow-up X along the point x, the point gets replaced by the exceptional divisor E, which we can view as a line bundle, under the line bundle-divisor correspondence.

Recall we had the short exact sequences of sheaves:

$$0 \to L \otimes \mathcal{I}_x \to L \to L_x \to 0$$
$$0 \to L \otimes \mathcal{I}_{x,y} \to L \to L_x \oplus L_y \to 0$$
$$0 \to L \otimes \mathcal{I}_x^2 \to L \otimes \mathcal{I}_x \to L_x \otimes (T^*_{\mathbb{C}}X)^{1,0} \to 0$$

We can blow up along the points x and y, as needed. In virtue of Lemma 5.38, we get:

$$0 \to \sigma^* L \otimes \mathcal{O}(-E) \to \sigma^* L \to \sigma^* L|_E \to 0$$
$$0 \to \sigma^* L \otimes \mathcal{O}(-E_1 - E_2) \to \sigma^* L \to \sigma^* L|_{E_1 \cup E_2} \to 0$$
$$0 \to \sigma^* L \otimes \mathcal{O}(-2E) \to \sigma^* L \otimes \mathcal{O}(-E) \to \big(\sigma^* L \otimes \mathcal{O}(-E)\big)\big|_E \to 0$$

where $E = E_1$ and E_2 are the exceptional divisors of the blow-up of X along x and y respectively. Note that we are abusing notation and using $\sigma : \widehat{X} \to X$ to denote both the blow-up at x and the blow-up at x and y. So, we can consider the "blown-up" conditions:

(I_B) $H^1\big(\widehat{X}, \sigma^* L \otimes \mathcal{O}(-E)\big) = 0$,

(II_B) $H^1\big(\widehat{X}, \sigma^* L \otimes \mathcal{O}(-E_1 - E_2)\big) = 0$, and

(III_B) $H^1\big(\widehat{X}, \sigma^* L \otimes \mathcal{O}(-2E)\big) = 0$ for all $x \in X$.

Proposition 8.19. *Let (X^n, J) be a closed complex manifold of dimension $n \geq 2$. Then conditions $(I_B) - (III_B)$ imply $(I_H) - (III_H)$.*

Proof. Let us show that (I_B) and (II_B) imply (I_H) and (II_H) respectively. The blow-up map $\sigma : \widehat{X} \to X$ induces a commutative diagram in cohomology

$$\begin{array}{ccccc} H^0(X, L^k) & \longrightarrow & L(x) & \longrightarrow & H^1\big(X, \mathcal{I}_x(L)\big) \\ \downarrow{\scriptstyle \sigma^*} & & \downarrow{\scriptstyle \cong} & & \downarrow \\ H^0(\widehat{X}, \sigma^* L) & \longrightarrow & H^0(E, \mathcal{O}_E) \otimes L(x) & \longrightarrow & H^1\big(\widehat{X}, \sigma^* L \otimes \mathcal{O}(-E)\big)\ . \end{array}$$

The claim is equivalent to showing that the left vertical maps is surjective [4]. This map is given by the pullback on sections of L. A section $s \in H^0(\widehat{X}, \sigma^* L)$ restricts to a holomorphic section supported in $\widehat{X} \setminus E \cong X \setminus \{x\}$. Since $n \geq 2$, points have at least codimension 2, so Hartogs' Principle, Theorem 1.16, implies that $\widetilde{s} := \sigma_* \left(s|_{\widehat{X} \setminus E}\right)$ extends to a section of L, i.e. $\widetilde{s} \in H^0(X, L)$. Replacing E with $E_1 + E_2$ shows that (II_B) implies (II_H).

Finally, for the injective differential condition, we have

$$\begin{array}{ccccc} H^0(X, \mathcal{I}_x(L)) & \longrightarrow & (T^*_{\mathbb{C}} X)^{1,0} \otimes L(x) & \longrightarrow & H^1\big(X, \mathcal{I}_x^2(L)\big) \\ \downarrow{\scriptstyle \sigma^*} & & \downarrow{\scriptstyle \sigma^*} & & \downarrow \\ H^0\big(\widehat{X}, \sigma^* L \otimes \mathcal{O}(-E)\big) & \longrightarrow & H^0\big(E, \mathcal{O}_E(-E)\big) \otimes \sigma^* L(x) & \longrightarrow & H^1\big(\widehat{X}, \sigma^* L \otimes \mathcal{O}(-2E)\big)\ . \end{array}$$

By Corollary 5.46, we have $\mathcal{O}_E(E) = \mathcal{O}(-1)$, so

$$H^0\big(E, \mathcal{O}_E(-E)\big) \cong H^0\big(\mathbb{P}(T_x^{1,0} X), \mathcal{O}(-1)\big) \cong (T_{\mathbb{C}} X)^{1,0}\ ,$$

[4] In fact, the induced maps σ^* are injective since the blow-up map is surjective, so we are proving the the map $\sigma^* H^0(X, L) \to H^0(\widehat{X}, \sigma^* L)$ is an isomorphism.

so the middle vertical map is an isomorphism. And by the same argument as above, the left map is also an isomorphism. □

We have rephrased the embedding conditions in terms of the vanishing of some line bundle cohomology groups. To finish the proof of the Kodaira embedding theorem we need the following lemma:

Lemma 8.20. *Let $\sigma : \hat{X} \to X$ be the blow-up of X at finitely many points $x_1, \ldots, x_p$ with exceptional divisors $E_i = \sigma^{-1}(x_i)$, and $L \to X$ a positive line bundle. For any holomorphic line bundle $M \to X$ and $n_i > 0$, the bundle $\sigma^*(M \otimes L^k) \otimes \mathcal{O}(-\sum n_i E_i)$ is positive for $k \gg 0$.*

The idea of the proof is the following. First, we need to show that if L is a positive line bundle, then $L^k \otimes M$ is positive for $k \gg 0$ large enough. Second, the pullback of a positive line bundle is only semi-positive since $c_1(\sigma^* L)$ will vanish when restricted to TE, so we need to compensate with something positive supported on E:

Proof. First, let us construct a metric on $\mathcal{O}(-\sum n_i E_i)$ such that the curvature of the Chern connection is semi-positive near E_i and positive in tangential directions of E_i.

Choose $x_i \in U_i \cong B_\varepsilon$ neighbourhoods with $U_i \cap U_j = \emptyset$. Consider their pullbacks to $\hat{X}$, $\hat{U}_i = \hat{X}|_{U_i} \cong \{(z, l) \in B_\varepsilon \times \mathbb{CP}^{n-1} \mid z \in l\} \to \mathbb{CP}^{n-1}$ and $\mathcal{O}(E_i)|_{\hat{U}_i} = p^* \mathcal{O}(-1)$. So, it suffices to pull back the standard Fubini-Study metric on $\mathcal{O}(1)$, since $F_\nabla = -2\pi i \; p^*(\omega_{FS})$. The desired metric follows by using a partition of unity subordinate to $\{U_i\}$.

Finally, let $n_i \geq 0$, and α, β real $(1,1)$-forms on X, with α positive and set $\hat{F} = \sigma^*(k\alpha + \beta) + \sum_i n_i F_{\nabla_i}$. It is straightforward to check that $\hat{F}$ is positive, since F_{∇_i} is positive along TE_i $\sigma^*(k\alpha + \beta)|_{TE_i} = 0$; and $\sigma^*(k\alpha + \beta)$ is positive for $k \gg 0$ elsewhere since α is positive.

The claim follows by taking α and β such that $[\alpha] = c_1(L)$ and $[\beta] = c_1(M)$. □

We can now finally give a proof of the Kodaira embedding theorem:

Proof of Theorem 8.16. In the notation above, let $k \gg 0$ large enough so the line bundles

$$\begin{aligned} M_1 &= \sigma^*(L^k \otimes K_X^*) \otimes \mathcal{O}(-nE) \\ M_2 &= \sigma^*(L^k \otimes K_X^*) \otimes \mathcal{O}(-nE_1 - nE_2) \\ M_3 &= \sigma^*(L^k \otimes K_X^*) \otimes \mathcal{O}\big((-n-1)E\big) \end{aligned}$$

are all positive, in virtue of Lemma 8.20. By Proposition 5.45, we have isomorphisms

$$\begin{aligned} M_1 &\cong \sigma^*(L^k) \otimes K_{\hat{X}}^* \otimes \mathcal{O}(-E) \\ M_2 &\cong \sigma^*(L^k) \otimes K_{\hat{X}}^* \otimes \mathcal{O}(-E_1 - E_2) \\ M_3 &\cong \sigma^*(L^k) \otimes K_{\hat{X}}^* \otimes \mathcal{O}(-2E) \; . \end{aligned}$$

Since the line bundles M_i are positive, the Kodaira Vanishing Theorem 8.9 implies

$$H^1\big(\widehat{X}, \sigma^*L \otimes \mathcal{O}(-E)\big) \cong H^1\Big(\big(\sigma^*(L^k) \otimes \mathcal{O}(-E) \otimes K^*_{\widehat{X}}\big) \otimes K_{\widehat{X}}\Big)$$
$$\cong H^1(\widehat{X}, M_1 \otimes K_{\widehat{X}}) = 0 \,, \tag{28a}$$
$$H^1\big(\widehat{X}, \sigma^*L \otimes \mathcal{O}(-E_1 - E_2)\big) \cong H^1(\widehat{X}, M_2 \otimes K_{\widehat{X}}) = 0 \,, \tag{28b}$$
$$H^1\big(\widehat{X}, \sigma^*L \otimes \mathcal{O}(-2E)\big) = H^1(\widehat{X}, M_3 \otimes K_{\widehat{X}}) = 0 \,. \tag{28c}$$

These are precisely the vanishing conditions $(I_B) - (III_B)$ for the line bundle L^k. Therefore, the pluricanonical map ϕ_{L^k} gives a projective embedding for k large enough, as needed. □

A first consequence of the Kodaira Embedding Theorem, is that the projectivity of a compact Kähler manifold is encoded in its Kähler cone:

Proposition 8.21. *A compact Kähler manifold (X, J, ω) is projective if and only if*

$$\mathcal{K}_X \cap Im(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R})) \neq \emptyset \,.$$

Proof. Assume $[\alpha] \in \mathcal{K}_X \cap \mathrm{Im}\big(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R})\big)$. Since $[\alpha] \in \mathcal{K}_X$, we have $[\alpha] \in H^{1,1}(X,\mathbb{Z})$ and the Lefschetz $(1,1)$-Theorem 7.17 implies there is a holomorphic line bundle L with $c_1(L) = [\alpha] > 0$. Thus, L is a positive line bundle on X, and by the Kodaira Embedding Theorem 8.16, gives rise to a projective embedding of X.

Conversely, given an embedding $\phi : X \to \mathbb{CP}^N$, the line bundle $L = \phi^*\big(\mathcal{O}(1)\big)$ satisfies $c_1(L) \in \mathcal{K}_X \cap \mathrm{Im}\big(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R})\big)$. □

Corollary 8.22. *Every compact Kähler manifold (X, J, ω) such that $H^{0,2}(X) = 0$ is projective.*

Proof. If $H^{0,2}(X) = 0$, we have $H^2(X,\mathbb{R}) \cong H^{1,1}(X,\mathbb{R})$, and so a multiple of the Kähler class $[\omega]$ will be an integral class. □

A surprising consequence of this statement is

Corollary 8.23. *Let (X^n, J, ω) be compact Kähler strict weak Calabi–Yau manifold of dimension $n \geq 3$. Then X is projective.*

It is worth noting that this is false for $\dim_{\mathbb{C}} X = 2$. In fact, in this case, "most" strict Calabi–Yau manifolds are not projective.

We have shown that every complex manifold with a positive line bundle can be embedded into a projective space of sufficiently large dimension, $\mathbb{CP}^N$. A natural question is then how large N needs to be. We have

Theorem 8.24. *Let X be a projective manifold of dimension n. Then $X \subseteq \mathbb{CP}^{2n+1}$.*

Note that, in general, the embedding $\phi : X \to \mathbb{CP}^{2n+1}$ will not arise as a pluricanonical map. To prove the theorem, it is convenient to introduce the secant variety. Let $X \subseteq \mathbb{CP}^N$. The secant map is defined as

$$\begin{aligned} s : X \times X \setminus \Delta &\to \mathbb{CP}^N \\ (x, y) &\mapsto [\overline{xy}] \end{aligned}$$

where $\overline{xy}$ is the line joining x and y and Δ is the diagonal embedding $X \to X \times X$. The closure of the lines in the image of the secant map is called the secant variety, denoted by $\mathrm{Sec}(X)$.

Lemma 8.25. *The secant variety* $\mathrm{Sec}(X)$ *is an analytic set of dimension at most twice the dimension of* X *plus one.*

Prof (Sketch). For $n \geq 1$, consider the 3-alternating map

$$\begin{aligned} \bigwedge^3 : \mathbb{CP}^N \times \mathbb{CP}^N \times \mathbb{CP}^N &\to \mathbb{P}\left(\bigwedge^3 \mathbb{C}^{N+1}\right) \\ ([a], [b], [c]) &\mapsto [a \wedge b \wedge c] \ . \end{aligned}$$

The secant variety Sec(X), away from the diagonal, is the projection to the third component of the vanishing locus of $\bigwedge^3$ restricted to $(X \times X \setminus \Delta) \times \mathbb{CP}^N$, which is analytic since both the projection map and $\bigwedge^3$ are analytic maps. The relation $\dim \mathrm{Sec}(X) = 2 \dim X + 1$ follows, since the fibre of the secant map is finite unless the line $\overline{xy}$ is already contained in X.

Near the diagonal, the secant map s converges to the "tangent line" at x in the direction of y, so the closure of the image s corresponds to adding the set $(X, X, \mathbb{CP}(TX))$, which is again analytic, so Sec(X) is can be shown to be everywhere analytic [5]. □

Proof (of Theorem 8.24). Consider $X \subseteq \mathbb{CP}^N$ with $N \geq 2 \dim X + 2$. We need to show that X can be embedded in $\mathbb{CP}^{N-1}$. By the previous lemma $N > \dim \mathrm{Sec}(X)$, so there exists at least a point p such that $p \notin \mathrm{Sec}(X)$. Take $H \cong \mathbb{CP}^{N-1} \subseteq \mathbb{CP}^N$ with $p \notin H$ and consider the projection map

$$\begin{aligned} \pi_p : X &\to H \\ x &\mapsto \overline{xp} \ \cap H \ . \end{aligned}$$

Then π_p is an injective holomorphic map since $p \notin$ Sec(X). But the differential of an injective holomorphic map has maximal rank, so Theorem 1.6 implies π_p is a biholomorphism when restricted to the image. □

One can show that the bound $N = 2 \dim X + 1$ is sharp, i.e. that there exist projective manifolds of dimension n that cannot be embedded in $\mathbb{CP}^{2n}$.

[5] This approach to deal with the diagonal corresponds to the classical "italian" approach to the problem. The modern "francogermran" approach consists of replacing $X \times X$ by the Hilbert scheme of two points, $Hilb_2(X)$, for which the secant map can be defined unambiguously.

9 Automorphisms in complex geometry

Let us now study the automorphism group of different structures in complex geometry. We will first focus on the group of automorphisms of holomorphic vector bundles $E \to X$, and then study the group of biholomorphisms of complex manifolds (X, J). Throughout, we take X to be a closed complex manifold, although we do not impose Kähler or projective conditions.

9.1 Automorphisms of holomorphic bundles

Given a complex vector bundle $E \to X$ of rank r, there is a natural associated $\mathrm{GL}(r, \mathbb{C})$-fibre bundle, called the frame bundle of E, whose transition functions are the same as the ones for E. We will denote it by $\mathrm{GL}(E) \to X$ [6].

The space of sections of the frame bundle $\mathcal{A}^0_X\big(\mathrm{GL}(E)\big)$ carries a natural group structure given by pointwise composition (matrix multiplication). Therefore, it acts naturally on E and its sections. Given a holomorphic vector bundle E, we are interested in computing the subgroup of $\mathcal{A}^0_X\big(\mathrm{GL}(E)\big)$ that preserves the holomorphic structure of E, denoted $\mathrm{Aut}(X, E)$. The main goal of this section is to prove that

Theorem 9.1. *The group* $\mathrm{Aut}(X, E)$ *is a complex Lie group of (complex) dimension* $H^0(X, \mathrm{End}(E))$.

The full proof of the theorem involves mild technical analytical considerations that we will not discuss in detail. Instead, we black-box the following result:

Theorem 9.2. *The space* $\mathcal{G} := \mathcal{A}^0_X\big(\mathrm{GL}(E)\big)$ *of a smooth vector bundle* $E \to X$ *is a Fréchet Lie group, with Lie algebra* $\mathcal{A}^0_X\big(\mathrm{End}(E)\big)$.

As the reader might have realised, the main challenge of proving the theorem above is having the appropriate notion of what an infinite-dimensional manifold should be. Once that is sorted, one may use the usual (pointwise) exponential map

$$\begin{aligned} \exp_t : \mathrm{End}(r, \mathbb{C}) &\to \mathrm{GL}(r, \mathbb{C}) \\ A &\mapsto \sum_{k \geq 0} \frac{A^k}{k!} t^k \end{aligned}$$

to show that $\mathcal{G} = \mathcal{A}^0_X\big(\mathrm{GL}(E)\big)$ carries the required manifold structure. In fact, for our purposes, it suffices to work with the Sobolev version $\mathcal{G}^k = \mathcal{A}^0_X\big(\mathrm{GL}(E)\big)^k$ for some k large enough.

Let us proceed towards the proof of Theorem 9.1. First, we have,

Lemma 9.3. *The space* $H^0(X, \mathrm{End(E)})$ *is the (formal) tangent space of* $\mathrm{Aut}(X, E)$.

Proof. The group $\mathcal{G}$ acts naturally on E and its sections by pullback. In particular, this induces an action on $\overline{\partial}_E$. We are interested in understanding for which $g \in \mathcal{G}$, we get $\overline{\partial}_{g^*E} = \overline{\partial}_E$. For a

[6] The frame bundle $\mathrm{GL}(E)$ carries the additional structure of a principal bundle, which is key in understanding vector bundles and connections. We refer the interested reader to the classic books of Kobayashi and Nomizu "Foundations of Differential Geometry" for a detailed treatment of principal bundles.

section s, we have

$$g^* \left(\overline{\partial}_E\right)(s) = g^{-1}\left(\overline{\partial}_E(gs)\right) = g^{-1}\left(g\overline{\partial}_E s + (\overline{\partial}_E g)s\right) ,$$

where we identify $\overline{\partial}_E : \mathcal{A}^0(E) \to \mathcal{A}^{0,1}(E)$ with the induced operator on $\mathcal{A}^\bullet(\mathrm{GL}(E))$. If we linearise the above formula, for $g_t = \mathrm{Id} + th + O(t^2)$, we get the condition $\overline{\partial}_E h = 0$, as needed. □

Therefore, it follows that $\mathrm{Aut}(X, E)$ is a finite-dimensional subgroup of $\mathrm{Diff}(X)$. However, since $\overline{\partial}_E$ is a continuous operator in the induced topology (try to think why), $\mathrm{Aut}(X, E)$ is a closed subgroup of a Lie group, so it is an embedded subgroup of the diffeomorphism group.

All that remains to prove is that $\mathrm{Aut}(X, E)$ naturally carries a complex Lie group structure. The following proposition allows us to reduce the question to the Lie algebra.

Proposition 9.4. *Let G be a connected Lie group, and assume its Lie algebra $\mathfrak{g}$ carries a complex structure J. Then, there is a natural almost complex structure $\hat{J}$ on G, given by*

$$\widehat{J}_g(\xi) \coloneqq J(\xi g^{-1})g$$

for $g \in G$ and $\xi \in T_g G$. The following are equivalent

(i) $(G, \widehat{J})$ *is a complex Lie group,*

(ii) the complex structure $\widehat{J}$ on G is Ad*-invariant, i.e.* $\widehat{J}(\mathrm{Ad}_g(\xi)) = \mathrm{Ad}_g\left(\widehat{J}(\xi)\right)$,

(iii) the Lie bracket is J-bilinear, $J[\xi, \chi] = [J\xi, \chi] = [\xi, J\chi]$ *for all* $\xi, \chi \in \mathfrak{g}$.

Proof. The equivalence between (*ii*) and (*iii*) follows since the Lie bracket is the differential of the adjoint map Ad_g, and G is connected.

Now, note that right multiplication map $r_h : g \mapsto gh$ is automatically holomorphic, since

$$\widehat{J}_{gh}(\xi h) = J\big(\xi h (gh)^{-1}\big)gh = J(\xi g^{-1})gh = \widehat{J}_g(\xi)h ,$$

so $r_h^* \widehat{J} = \widehat{J}$, as needed.

Next, we claim that left multiplication $l_h : g \mapsto hg$ is holomorphic if and only if J is Ad-invariant. Indeed, we have

$$\widehat{J}_{hg}(h\xi) - h\widehat{J}_g(\xi) = J(h\xi g^{-1}h^{-1})hg - hJ(\xi g^{-1})g = \left[J(h\xi g^{-1}h^{-1}) - hJ(\xi g^{-1})h^{-1}\right]hg .$$

Therefore, (*i*) implies (*ii*).

Let us prove that (*iii*) implies (*i*). First, let us check that the Nijenhuis tensor vanishes pointwise. Since it is a tensor, it is enough to check that it vanishes for left-invariant vector fields $(l_h)_* \xi_g = \xi_{hg}$. Note that, by our definition of the almost complex structure $\widehat{J}$, they satisfy

$$\widehat{J}\xi_g = (J\xi)_g \qquad\qquad [\xi_g, \eta_g] = ([\xi, \eta])_g .$$

The Nijenhuis tensor on left invariant vector fields is

$$\begin{aligned}
N_{\widehat{J}}(\xi_g, \eta_g) &= [\xi_g, \eta_g] + \widehat{J}([\widehat{J}\xi_g, \eta_g] + [\xi_g, \widehat{J}\eta_g]) - [\widehat{J}\xi_g, \widehat{J}\eta_g] \\
&= [\xi_g, \eta_g] + \widehat{J}([(J\xi)_g, \eta_g] + [\xi_g, (J\eta)_g]) - [(J\xi)_g, (J\eta)_g] \\
&= [\xi, \eta]_g + \big(J([(J\xi), \eta] + [\xi, (J\eta)])\big)_g - [(J\xi), (J\eta)]_g \\
&= \big([\xi, \eta] + J([(J\xi), \eta] + [\xi, (J\eta)]) - [(J\xi), (J\eta)]\big)_g \,.
\end{aligned}$$

Thus, if the Lie bracket is J-bilinear, $N_{\widehat{J}}$ vanishes, as needed. Left and right multiplication are holomorphic by the previous argument. Let us argue that the inverse map is holomorphic. Since multiplication is a submersion, the preimage of the neutral element $e \in G$ is a complex submanifold of $G \times G$, which coincides with the graph of the inverse map $g \mapsto g^{-1}$. □

Therefore, we have a complex Lie group of automorphisms $\mathrm{Aut}(E)$.

Lemma 9.5.

(i) The centre of $\mathrm{Aut}(E)$ *contains a copy of* $\mathbb{C}^*$*, given by constant rescalings.*

(ii) If $E \cong E_1 \oplus E_2$ *splits holomorphically, then* $(\mathbb{C}^*)^2 \subseteq \mathrm{Aut}(E)$*.*

Proof. Note that there is a natural (therefore holomorphic) splitting $\mathrm{End}(E) \cong \underline{\mathbb{C}} \oplus \mathrm{End}^0(\mathrm{E})$. The first term corresponds to the trace-part of the endomorphisms, and a global trivialising is given by constant rescalings.

Similarly, given E_1 and E_2 holomorphic bundles of ranks r_1 and r_2 respectively, the family of endomorphisms

$$\Phi_{\lambda,\mu} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} \mu r_2 & 0 \\ 0 & -\mu r_1 \end{pmatrix} \qquad \mu, \lambda \in \mathbb{C}$$

spans a two-dimensional abelian subalgebra of holomorphic sections of $\mathrm{End}(E) = \underline{\mathbb{C}} \oplus \mathrm{End}^0(E)$. □

We would like to find a converse of (ii) that guarantees that the vector bundle splits whenever $\dim H^0(X, \mathrm{End}_0(E)) \neq 0$. We have

Proposition 9.6. *Consider* $E \to X$ *a holomorphic vector bundle of rank two, with* $0 \neq \Phi \in H^0(X, \mathrm{End}^0(E))$*. Then, there exist two holomorphic line bundles* L_1, L_2 *such that* $E = L_1 \oplus L_2$*.*

The proof is not particularly complicated, but we omit the details. Consider the minimal polynomial of Φ, $\chi_\Phi = t^2 + \det(\Phi)$. Then $\det(\Phi)$ is a holomorphic function, so either it has isolated zeroes or is everywhere zero. In the former case, we have $\det \Phi = -\lambda^2$ and

$$\Phi = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} .$$

Then $L_i = L_\pm := \ker(\mathrm{Id} \pm \frac{1}{\lambda}\Phi)$ gives the requires decomposition of E away from the vanishing locus $Z(\det(\Phi))$. One then has to argue that this decomposition extends over $Z(\det(\Phi))$.

If $\det(\Phi) = 0$, the endomorphism Φ is nilpotent, so $L = \ker(\Phi) \subseteq \operatorname{Im}(\Phi)$, and we get the short exact sequence

$$0 \to L \to E \xrightarrow{\Phi} L \to 0 ,$$

where locally Φ is of the form $\begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$. We claim that λ is globally well defined, so it defines a map $E/L \to E$, giving the required splitting.

For higher rank bundles, the argument above fails, and there might not exist subbundles E_i decomposing E when $H^0(X, \operatorname{End}(E)) \neq 0$, as the "ranks" of the subbundles E_i might not be locally constant. However, there will exist a splitting of E in terms of coherent sheaves $\mathcal{E}_i$. Therefore, we make the definition

Definition 9.7. A holomorphic bundle $E \to X$ is called *reducible* whenever $H^0(X, \operatorname{End}^0(E)) \neq 0$. We call it *irreducible* otherwise.

9.2 Automorphisms of complex manifolds

Let us now consider the automorphisms of a complex manifold (X, J); the group

$$\operatorname{Aut}(X, J) \coloneqq \{f : X \to X \,|\, f \text{ is a biholomorphism}\} .$$

Analogously to the previous case, we prove that

Theorem 9.8. *The group* $\operatorname{Aut}(X, J)$ *is a complex Lie group of (complex) dimension* $H^0(X, \tau_X)$*.*

The proof strategy is the same as in the previous case, so we omit some details to avoid repetition. The automorphism group $\operatorname{Aut}(X, J)$ is a subgroup of the diffeomorphism group. As before, we claim

Theorem 9.9. *The diffeomorphism group* $\operatorname{Diff}(X)$ *of a smooth manifold* X *is a Fréchet Lie group. Its Lie algebra is the space of smooth vector fields* $\mathfrak{X}(X) = \mathcal{A}^0_X(TX)$*.*

Therefore, we compute which vector fields preserve the almost complex structure. We have

Lemma 9.10. *The space* $H^0(X, \tau_X) \subseteq \mathfrak{X}(X)$ *is the (formal) tangent space of* $\operatorname{Aut}(X, J)$*.*

Proof. We want to characterise diffeomorphisms $\phi \in \operatorname{Diff}(X)$ that commute with I under the push forward action: $\phi_* \circ I = I \circ \phi_*$. Working in coordinates $\{z_i, \overline{z_i}\}$, this condition is equivalent to

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \partial_{z_j}\phi_i & \partial_{\overline{z_j}}\phi_i \\ \partial_{z_j}\overline{\phi_i} & \overline{\partial_{z_j}\phi_i} \end{pmatrix} = \begin{pmatrix} \partial_{z_j}\phi_i & \partial_{\overline{z_j}}\phi_i \\ \partial_{z_j}\overline{\phi_i} & \overline{\partial_{z_j}\phi_i} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

Which is equivalent to $\overline{\partial}\phi = 0$. Consider a 1-parameter family of diffeomorphisms, $\phi_t = \operatorname{Id} + (v + \overline{v})\frac{t}{2} + O(t^2)$ with $v \in \mathcal{A}^0(T^{1,0}_{\mathbb{C}}X)$. The claim follows by linearising $\overline{\partial}\phi_t = 0$. □

Therefore, by the same reasoning as in the bundle case, $\operatorname{Aut}(X, J)$ is a finite-dimensional Lie group. The proof of Theorem 9.8 is complete, up to showing that

Lemma 9.11. *Let (X, J) be a complex manifold. Then $H^0(X, \tau_X)$ is naturally a complex Lie algebra, for which the Lie bracket is complex bilinear.*

This final computation is left to the reader (cf. Exercise 61).

We have shown that every closed complex manifold has an associated complex Lie group $\operatorname{Aut}(X, J)$. We conclude this discussion by studying some examples of $\operatorname{Aut}(X, J)$, or at least part of it. If the complex structure is understood, we will simply write $\operatorname{Aut}(X)$.

Theorem 9.12. *For the complex projective space $\mathbb{CP}^n$, we have a short exact sequence of groups*

$$0 \to \mathbb{C}^* \to \mathrm{GL}(n+1, \mathbb{C}) \to \operatorname{Aut}(\mathbb{CP}^n) \to 0\ ,$$

where the first map is given by the diagonal embedding $1 \to \mathrm{Id}_{n+1}$.

The group $\mathrm{GL}(n+1, \mathbb{C})/\mathbb{C}^*$ is called the projective general linear group, denoted by $\mathrm{PGL}(n+1, \mathbb{C})$.

Proof. It is clear that the action of $\mathrm{GL}(n+1, \mathbb{C})$ on $\mathbb{C}^{n+1}$ descends to a biholomorphic action of $\mathbb{CP}^n$, so we have a map $\mathrm{GL}(n+1, \mathbb{C}) \to \operatorname{Aut}(\mathbb{CP}^n)$. An element of the kernel must fix all lines in $\mathbb{C}^{n+1}$, so it must be a global rescaling $\mu \, \mathrm{Id} \in \mathrm{GL}(n+1, \mathbb{C})$ for $\mu \in \mathbb{C}^*$.

Using the Euler sequence, it follows easily that the connected component of the identity of $\operatorname{Aut}(\mathbb{CP}^n)$, denoted $\operatorname{Aut}^0(\mathbb{CP}^n)$, is precisely $\mathrm{PGL}(n+1, \mathbb{C})$ (cf. Exercise 63). Thus, we need to prove that the discrete subgroup $G = \operatorname{Aut}(\mathbb{CP}^n)/\operatorname{Aut}^0(\mathbb{CP}^n)$ is trivial.

The idea is the following: the automorphism group $\operatorname{Aut}(\mathbb{CP}^n)$ acts naturally on $\operatorname{Pic}(\mathbb{CP}^n) \cong \mathbb{Z}$, so it must map generators to generators, and since it must preserve orientation, we have $f^*\mathcal{O}(1) \cong \mathcal{O}(1)$ for all $f \in \operatorname{Aut}(\mathbb{CP}^n)$. Therefore, the space of sections $H^0\big(\mathbb{CP}^n, \mathcal{O}(1)\big)$ inherits an action by $\operatorname{Aut}(\mathbb{CP}^n)$. But, $H^0\big(\mathbb{CP}^n, \mathcal{O}(1)\big)$ is a complex vector space of dimension $n+1$, so there is an induced map $\Phi : \operatorname{Aut}(\mathbb{CP}^n) \to \mathrm{PGL}(n+1, \mathbb{C})$, and it suffices to prove that the map Φ is injective.

Let $\{s_i\}$ be a basis of $H^0\big(\mathbb{CP}^n, \mathcal{O}(1)\big)$ and consider $f \in \ker \Phi$, so $f^*(s_i) = s_i$ for all $i = 0, \ldots, n$. But the pluricanonical map associated to $\phi_{\{s_i\}}$ is an automorphism of $\mathbb{CP}^n$ (check it!), which fits in the commutative square

$$\begin{array}{ccc} \mathbb{CP}^n & \xrightarrow{\ f\ } & \mathbb{CP}^n \\ \big\downarrow{\scriptstyle \phi_{\{s_i\}}} & & \big\downarrow{\scriptstyle \phi_{\{s_i\}}} \\ \mathbb{CP}^n & \xrightarrow{\ \mathrm{Id}\ } & \mathbb{CP}^n \end{array}$$

□

Similarly, we have

Theorem 9.13. *The automorphism group of the torus $\mathbb{T}^n \cong \mathbb{C}^n/\Gamma$ is $\operatorname{Aut}(\mathbb{T}^n) \cong \mathbb{T}^n \rtimes \operatorname{Aut}(\Gamma)$, where*

$$\operatorname{Aut}(\Gamma) := \big\{ g \in \mathrm{GL}(n, \mathbb{C}) \big|\ g(\Gamma) = \Gamma \big\} \subseteq \mathrm{GL}(n, \mathbb{C})\ .$$

Proof. First, let us show that $\mathrm{Aut}^0(\mathbb{T}^n) \cong \mathbb{T}^n$. We clearly have $\mathbb{T}^n \subseteq \mathrm{Aut}^0(\mathbb{T}^n)$, where $\mathbb{T}^n$ acts on itself by translations. But, we have

$$H^0(\mathbb{T}^n, \tau_{\mathbb{T}^n}) \cong H^0(\mathbb{T}^n, \mathcal{O}^n_{\mathbb{T}^n}) \cong \mathbb{C}^n ,$$

and the inclusion is actually an equality.

Thus, it suffices to prove $\mathrm{Aut}(\mathbb{T}^n)/\,\mathrm{Aut}^0(\mathbb{T}^n) \cong \mathrm{Aut}(\Gamma)$. Let f be a representative of a non-trivial class in $\mathrm{Aut}(\mathbb{T}^n)/\,\mathrm{Aut}^0(\mathbb{T}^n)$. Since the action by translations is (sharply) transitive, the map f must have a fixed point.

Consider $\widetilde{f} : \mathbb{C}^n \to \mathbb{C}^n$ the biholomorphic lift of f to its universal cover $\mathbb{C}^n$, with the convention that the fixed point gets mapped to zero. Then $\widetilde{f}$ satisfies that the map

$$z \to \widetilde{f}(z + \lambda) - \widetilde{f}(z)$$

is constant for any $\lambda \in \Lambda$. Thus, ∂f is periodic. Liouville's Theorem 1.13 implies ∂f is constant. Since $\widetilde{f}(0) = 0$, we see that $\widetilde{f} \in GL(n, \mathbb{C})$. Since $\widetilde{f}(\Lambda) \subseteq \Lambda$, and $(\widetilde{f})^{-1}(\Lambda) = \widetilde{f^{-1}}(\Lambda) \subseteq \Lambda$, we get $\widetilde{f} \in \mathrm{Aut}(\Lambda)$. □

Finally, from Exercise 64, it immediately follows that

Proposition 9.14. *Let X be a complex manifold with $\chi(X) < 0$. Then $\mathrm{Aut}(X)$ is a discrete group.*

In particular, we get

Corollary 9.15. *The automorphism group of a Riemann surface of genus $g \geq 2$ is discrete.*

The previous result relies only on the underlying topology of the manifold. Let us give two general results that involve the complex geometry of the manifold.

Proposition 9.16. *Let X be a complex manifold such that its canonical line bundle is positive. Then $\mathrm{Aut}(X)$ is a discrete group.*

This gives an alternate proof to Corollary 9.15.

Proof. By Theorem 9.8, it suffices to show that $H^0(X, \tau_X) = 0$. By Serre duality, we have

$$H^0(X, \tau_X) \cong H^n(X, \tau_X^* \otimes K_X)^* \cong H^n(X, \Omega^1_X \otimes K_X)^* .$$

The latter vanishes by the Kodaira Vanishing Theorem 8.9, since K_X is positive. □

Complex manifolds with positive canonical line bundle are called of *general type*. Conversely, if their canonical line is negative, they are called *Fano*. By the Kodaira embedding theorem 8.16, both classes are automatically projective. Manifolds of general type are subclass of complex manifolds that remains an active topic of investigation, as showcased in dimension two by the Enriques–Kodaira classification [BPV84, Chapter VI].

In fact, the group $\mathrm{Aut}(X)$ is not only discrete but finite for manifolds of general type:

Theorem 9.17. *Let (X, J) be a closed complex manifold of general type. Then $\mathrm{Aut}(X, J)$ is finite.*

The original proof, due to Kobayashi, is quite elegant and not too complicated, but we skip it due to time constraints. The reader is encouraged to consult the original article of Kobayashi, [Kob59].

Finally, we have

Theorem 9.18. *Let X be a weak Calabi–Yau manifold. Then $\dim \mathrm{Aut}(X) = \dim H^0(X, \Omega_X^{n-1})$. In particular, $\mathrm{Aut}(X)$ is a discrete group for strict weak Calabi–Yau manifolds.*

Proof. We need to understand $H^0(X, \tau_X)$. Let Ω be a nowhere vanishing holomorphic section of $K_X \cong \mathcal{O}_X$. Then, have a map

$$\begin{aligned} \lrcorner\Omega : \mathcal{A}_X^0(\tau_X) &\to \Omega_X^{n-1} \\ f &\mapsto f \lrcorner \Omega \end{aligned}$$

that descends to an isomorphism $H^0(X, \tau_X) \cong H^0(X, \Omega_X^{n-1}) \otimes H^0(X, K_X)^*$. □

10 Deformations in complex geometry

We now turn to deformation problems. We begin with the case of holomorphic bundles, due to its (relative) simplicity, before turning to the case of complex manifolds.

10.1 Deformations of holomorphic vector bundles

We are interested in parametrising the collection of holomorphic structures on a given vector bundle and understanding what structure this collection of structures carries. As a motivating example, recall that back in Section 5.1, we showed that if a line bundle $L \to X$ carries a holomorphic structure, it carries $\mathcal{M}_L = H^1(X, \mathcal{O}_X)/\operatorname{im}(H^1(X, \mathbb{Z})$ many of them. If one further imposes that X is a Kähler manifold, then $\mathcal{M}_L$ is a complex torus of (complex) dimension given by the first Betti number $b^1(X)$.

We would like to extend this to the case of general holomorphic bundles. The general strategy is to view the corresponding moduli $\mathcal{M}_E$ as the zero set of some (infinite-dimensional) map, so $\mathcal{M}_E$ can be constructed via the implicit function theorem. In the case of holomorphic structures, we exploit Theorem 5.7.

We denote by $\mathcal{A}(X, E)$ the space of $\mathbb{C}$-linear operators $\overline{\partial}_E : \mathcal{A}_X^0(E) \to \mathcal{A}_X^{0,1}(E)$ they satisfies the generalised Leibniz rule

$$\overline{\partial}_E(fs) = \overline{\partial} f s + f \overline{\partial}_E s$$

for $f \in \mathcal{A}_X^0$ and $s \in \mathcal{A}_X^0(E)$.

Lemma 10.1. *The $\mathcal{A}(E)$ is an affine space modelled after $\mathcal{A}_X^{0,1}(\mathrm{End}(E))$.*

Proof. Given $\overline{\partial}_E \in \mathcal{A}(X,E)$, it is clear that $\overline{\partial}_E + \alpha \in \mathcal{A}(X,E)$. Conversely, given $\overline{\partial}_E, \overline{\partial}'_E \in \mathcal{A}(X,E)$, the generalised Leibniz rule implies that

$$(\overline{\partial}_E - \overline{\partial}'_E)(fs) = (df\ s + f\overline{\partial}_E s) - (df\ s + f\overline{\partial}'_E s) = f(\overline{\partial}_E - \overline{\partial}'_E)(s)\ ,$$

so $(\overline{\partial}_E - \overline{\partial}'_E)$ is function-linear and so can be identified with an element in $\mathcal{A}^{0,1}_X(\mathrm{End}(E))$. □

The characterisation of holomorphic structures of Theorem 5.7 implies

Proposition 10.2. *Let (E, ∂_E) be a holomorphic structure on E. An element $\overline{\partial}'_E = \overline{\partial}_E + \alpha \in \mathcal{A}(E)$ defines a holomorphic structure if and only if $\alpha \in \mathcal{A}^1_X(\mathrm{End}(E))$ satisfies*

$$\overline{\partial}_E \alpha + \frac{1}{2}[\alpha, \alpha] = 0\ . \tag{29}$$

Equation (29) is known as the Maurer–Cartan equation.

Proof. The holomorphicity condition is equivalent to $\left(\overline{\partial}'_E\right)^2 = \left(\overline{\partial}_E + \alpha\right)^2 = 0$. Expanding out for $\beta \in \mathcal{A}^k_X(E)$, we have

$$\begin{aligned}\left(\overline{\partial}_E + \alpha\right)^2(\beta) &= \left(\overline{\partial}_E + \alpha\right)\left(\overline{\partial}_E\beta + \alpha\wedge\beta\right)\\ &= \overline{\partial}^2_E\beta + \overline{\partial}_E(\alpha\wedge\beta) + \alpha\wedge\overline{\partial}_E\beta + \frac{1}{2}[\alpha,\alpha]\wedge\beta\\ &= \left(\overline{\partial}_E\alpha + \frac{1}{2}[\alpha,\alpha]\right)(\beta)\ .\end{aligned}$$

□

Notice that the space of solutions to the Maurer–Cartan Equation (29) is infinite dimensional, due to the following naive symmetry reason.

Consider $\mathcal{G} = \mathcal{A}^0_X\big(\mathrm{GL}(E)\big)$, the group of linear isomorphisms of the complex bundles E from the previous section. Then the condition $\overline{\partial}^2_E = 0$ is invariant under the action of $\mathcal{G}$:

$$g^*(\overline{\partial}^2_E) = \left(g^{-1}\circ\overline{\partial}_E\circ g\right)^2 = g^{-1}\circ\overline{\partial}^2_E\circ g = 0\ .$$

The tangent space of the $\mathcal{G}$- orbit at a point $\overline{\partial}_E$ is given by

$$T_{\overline{\partial}_E}\mathcal{G} = \left\{\overline{\partial}_E g \;\middle|\; g\in\mathcal{G}\right\} \subseteq \mathcal{A}(E)\ .$$

Since the $\mathcal{G}$-orbit provides "trivial" solutions to the Maurer–Cartan, we would like to study solutions to (29) modulo $\mathcal{G}$. Therefore we define the moduli space of holomorphic structures on E to be $\mathcal{M}_E \coloneqq \Phi^{-1}(0)/\mathcal{G}$, where Φ is the map

$$\begin{aligned}\Phi : \mathcal{A}(X,E) &\to \mathcal{A}^{0,2}_X(E)\\ \overline{\partial}_E &\mapsto \overline{\partial}^2_E\ .\end{aligned}$$

Alternatively, we can consider the space $\mathcal{B} \coloneqq \mathcal{A}(X,E)/\mathcal{G}$ and the projection of Φ. However, it is not hard to see that the space $\mathcal{B}$ is quite pathological, as it will not be Hausdorff in general. Therefore, one usually requires the holomorphic structure $(E, \overline{\partial}_E)$ to satisfy what is called a "stability" condition, to remove the points in $\mathcal{A}(X,E)$ that give rise to the non-Hausdorff phenomena in $\mathcal{B}$.

Let us illustrate with a finite-dimensional example. Consider $\mathbb{C}^*$ acting on $\mathbb{C}^2$ by $\lambda \cdot (z, w) \mapsto (\lambda z, \lambda^{-1} w)$. There are three types of orbits: the origin, the complex lines $(0, w)$ and $(z, 0)$, and the smooth conics $zw = c \in \mathbb{C}^*$, and the quotient $\mathbb{C}^2/\mathbb{C}^*$ is not Hausdorff, since the the complex lines and the origin can not be separated on the quotient. However, if we define the smooth conics to be the stable objects, the quotient is simply $\mathbb{C}^*$.

This is essentially the start of geometric invariant theory (GIT) and the Kempf–Ness theorem, and our discussion corresponds to an infinite-dimensional version of this discussion. For an introduction to GIT and stability conditions, the reader is referred to the excellent notes of Richard Thomas on the topic, [Tho06].

Just for completeness, we include the suitable stability condition needed in the case of holomorphic bundles:

Definition 10.3. Let (E, ∂_E) a holomorphic bundle over a Kähler manifold (X, ω). The slope of E is defined as

$$\mu(E) := \frac{1}{\mathrm{rk(E)}} \int_X \Lambda c_1(E) \operatorname{vol}_g = \frac{1}{\mathrm{rk(E)}} \int_X c_1(E) \wedge \frac{\omega^{n-1}}{(n-1)!} .$$

A holomorphic bundle is called stable (resp. semi-stable) if, for any holomorphic subbundle $(F, \overline{\partial}_F)$, one has $\mu(F) < \mu(E)$ (resp. $\leq$).

Fortunately, if we want to study the deformation theory of $(E, \overline{\partial}_E)$, we can "avoid" the difficulty of the non-Hausdorff quotient by considering *versal* deformations. If T is a complex space with base point t_0 we say that a deformation of the holomorphic bundle $(E, \overline{\partial}_E) \to X$ parametrized by T is a holomorphic bundle $(\widetilde{E}, \overline{\partial}_{\widetilde{E}})$ over $Z \times T$ which restricts to $(E, \overline{\partial}_E)$ on $Z \times \{t_0\}$. Given a deformation over (T, t_0) and an analytic map of pointed spaces $(S, s_0) \to (T, t_0)$, we get an induced deformation over (S, s_0) by pullback. This notion carries over at the level of germs, i.e. we regard two spaces as being equivalent if there is an isomorphism between some neighbourhoods of their base points, and maps as being equivalent if they agree in such neighbourhoods. We say that a deformation of $(E, \overline{\partial}_E)$ parametrised by (T, t_0) is *versal* if any other deformation can be induced from it by a map, and that the deformation is *universal* if the map is unique.

Thus, in a way, we are aiming to construct a "cover" of the moduli space $\mathcal{M}_E$. The main result, due to Kuranishi, is that versal deformations always exist, and they are universal when $\operatorname{Aut}(E) \cong \mathbb{C}^*$. Let us give an idea of the proof of this result.

We argued that the orbit of a holomorphic structure under the gauge group $\mathcal{G}$ gives rise to trivial solutions. Therefore, infinitesimally, one should work in the L^2-complement of the tangent space to the orbit of the group $\mathcal{G}$. This corresponds to the space $\ker \overline{\partial}_E^*$.

Therefore, the vanishing locus near the origin of the map

$$\begin{aligned} \widehat{\Phi} : \mathcal{A}_X^{0,1}(\operatorname{End}(E)) &\to \mathcal{A}_X^{0}(\operatorname{End}(E)) \oplus \mathcal{A}_X^{0,2}(\operatorname{End}(E)) \\ \alpha &\mapsto \left(\overline{\partial}_E^* \alpha, \overline{\partial}_E \alpha + \frac{1}{2}[\alpha, \alpha]\right) \end{aligned}$$

should give rise to a versal deformation of $(E, \overline{\partial}_E)$, assuming $\widehat{\Phi}^{-1}(0)$ were finite-dimensional.

If we were in a finite-dimensional setting, we could try to appeal to the implicit function theorem (cf. Theorem 1.6) by checking whether $d\Phi$ satisfies the usual requirements. In the infinite-dimensional setting, one has a similar result, provided the derivative map $d\Phi$ is *Fredholm*, which amounts to being "almost" invertible. We give the theorem statement here, for completeness.

Theorem 10.4 (Banach Implicit Function Theorem, [Lan83, Thm. 2.1])**.** *Let X, Y be Banach manifolds and let $f : X \to Y$ be a Fredholm map: its derivative $\mathcal{D}_x f$ is a Fredholm linear operator for all x, so the vector spaces $K = \ker(\mathcal{D}_x f)$ and $C = \operatorname{coker}(\mathcal{D}_x f)$ are finite-dimensional.*

Fix a point $x_0 \in X$, with $y_0 = f(x_0)$. Let $L = \mathcal{T}_{x_0} f$ be the derivative of f at x_0. There are charts (U, κ) for X, $(V, \tilde{\kappa})$ for Y and a vector space B such that

$$\kappa : U \to B \oplus K \qquad\qquad \tilde{\kappa} : V \to B \oplus C\ ,$$

such that $\kappa(x_0) = 0$, $\tilde{\kappa}(x_0) = 0$ and the map $F = \tilde{\kappa} \circ f \circ \kappa^{-1} : B \oplus K \to B \oplus C$ is given by $F(b, n) = (L(b), \Theta(b, n))$ on an open neighbourhood $W \subseteq B \oplus K$, with $\Theta : W \to C$ a smooth map.

Then we have $f^{-1}(y_0) = \Theta^{-1}(0)$. If $C = 0$, then Θ is identically zero, and $f^{-1}(y_0)$ is diffeomorphic to an open neighbourhood of 0 in K, and therefore a smooth manifold of the dimension of the kernel.

We wish to apply Theorem 10.4 to construct a versal deformation of $(E, \overline{\partial}_E)$. We will not prove that the range of $d\widehat{\Phi}$ is closed, and will only compute its kernel and cokernel.

Lemma 10.5. *We have*

(i) $\ker(d\widehat{\Phi}) = H^1(\operatorname{End}(E))$,

(ii) $\operatorname{coker}(d\widehat{\Phi}) = H^0(\operatorname{End}(E)) \oplus H^2(\operatorname{End}(E)) \oplus \overline{\partial}_E^*\big(\mathcal{A}_X^{0,3}(\operatorname{End}(E))\big)$.

Proof. Clearly $d\widehat{\Phi}(\alpha) = \left(\overline{\partial}_E^* \alpha, \overline{\partial}_E \alpha\right)$. The claim now follows from the Hodge decomposition. □

Thus, the cokernel $\operatorname{coker}(d\Phi)$ is not finite-dimensional in dimensions $n \geq 3$. This a priori would prevent us from applying the Implicit Function Theorem 10.4. However, notice that the problem comes from the obstruction map Θ, taking image in an infinite-dimensional space. However, we have

Lemma 10.6. *Restricted to $\ker(d\widehat{\Phi})$, the image of $\widehat{\Phi}$ is $\overline{\partial}_E$-closed, so the obstruction map takes values in $(H^0 \oplus H^2)(X, \operatorname{End}(E))$.*

Proof. If $\alpha \in H^1(X, \operatorname{End}(E))$, we have $\overline{\partial}_E \alpha = 0$ by the Hodge decomposition. Thus, we have

$$\overline{\partial}_E \Phi(\alpha) = \overline{\partial}_E \left(\overline{\partial}_E \alpha + \frac{1}{2}[\alpha, \alpha] \right) = [\alpha, \overline{\partial}_E \alpha] = 0\ .$$

□

In particular, one can modify the Inverse Function Theorem slightly to prove

Proposition 10.7. *Let $(E, \overline{\partial}_E)$ be a holomorphic bundle with $H^2(X, \mathrm{End}(E)) = 0$. Then there exists a versal deformation of $(E, \overline{\partial}_E)$ of dimension $H^1(X, \mathrm{End}(E))$.*

The main reason this versal deformation family might not be universal (and the reason why we had to consider versal deformation at all to begin with) is that the map $\widehat{\Phi}$ does not "see" holomorphic automorphism of $(E, \overline{\partial}_E)$, so the moduli space $\mathcal{M}_E$ should look-like $\Phi^{-1}(0)/\operatorname{Aut}(E)$, which will not be Hausdorff in general since $\operatorname{Aut}(E)$ is never compact. More precisely, we have the following statement, from the work of Kuranishi:

Theorem 10.8 (Kuranishi)**.** *Let $(E, \overline{\partial}_E) \to X$ be a holomorphic vector bundle over a complex compact manifold. Then there is a holomorphic map Θ from a neighbourhood of 0 in $H^1(X, \mathrm{End}(E))$ to $H^2(X, \mathrm{End}_0(E))$, such that*

(i) the family $T = \Theta^{-1}(0)$ is a versal deformation of $(E, \overline{\partial}_E)$,

(ii) the 2-jet of Θ is given by the bracket $[\cdot, \cdot] : H^1(X, \mathrm{End}(E)) \times H^1(X, \mathrm{End}(E)) \to H^2(X, \mathrm{End}_0(E))$,

(iii) if $\operatorname{Aut}(E) \cong \mathbb{C}^*$, the versal deformation T is universal, and a neighbourhood of $(E, \overline{\partial}_E)$ in $\mathcal{M}_E$ is homeomorphic to the underlying space of T, and

(iv) if $\operatorname{Aut}(E)$ is reductive, the map Θ can be chosen to be $\operatorname{Aut}(E)$-equivariant, and a neighbourhood of of $(E, \overline{\partial}_E)$ in $\mathcal{M}_E$ is homeomorphic to the underlying space of $T/\operatorname{Aut}(E)$.

Let us define the *virtual dimension* of the moduli space $\mathcal{M}_E$ to be

$$\text{v-dim}(\mathcal{M}_E) = H^1(X, \mathrm{End}(E)) - H^0(X, \mathrm{End}(E)) - H^2(X, \mathrm{End}(E)) .$$

Given the construction of the versal deformation space for E, the virtual dimension represents the expected dimension of the moduli space $\mathcal{M}_E$. This is the dimension that $\mathcal{M}_E$ would have if it were a smooth manifold at $[E]$, with the deformation theory unobstructed (i.e. when $H^2(X, \mathrm{End}(E)) = 0$). In particular, it serves as a lower bound for the actual dimension of the moduli space.

The virtual dimension becomes of special interest in the case where $\dim_{\mathbb{C}}(X) \leq 2$, since all higher cohomology groups vanish, so $\text{v-dim}(\mathcal{M}_E) = -\chi(X, \mathrm{End}(E))$, and the Hirzebruch–Riemann–Roch theorem implies the virtual dimension is a topological invariant of E. In particular, we have

Proposition 10.9. *Let $E \to \Sigma_g$ be a holomorphic vector bundle of rank r over a Riemann surface of genus g. Then $\text{v-dim}(\mathcal{M}_E) = r^2(g-1)$.*

Proof. We need to compute $-\chi(\Sigma_g, \mathrm{End}(E))$. By the Hirzebruch–Riemann–Roch Theorem,

$$\begin{aligned} -\chi(\Sigma_g, \mathrm{End}(E)) &= -\int_{\Sigma_g} \mathrm{ch}(\mathrm{End}(E) \, \mathrm{Td}(\Sigma_g) \\ &= -\int_{\Sigma_g} \Big(\mathrm{rk}\left(\mathrm{End}(E)\right) + c_1(\mathrm{End}(E)\Big) \wedge \Big(1 + \frac{c_1(\Sigma_g)}{2}\Big) \\ &= -\int_X \frac{r^2 c_1(\Sigma_g)}{2} + c_1(\mathrm{End}(E)) = r^2(g-1) , \end{aligned}$$

where we used that $c_1(\mathrm{End}(E) = c_1(E) + c_1(E^*) = 0$ and $c_1(\Sigma_g) = 2 - 2g$ (cf. Exercise 53) in the last line. □

One can get a similar expression for complex surfaces, although in this case the virtual dimension will depend explicitly on the Chern classes of E.

10.2 Deformations of complex manifolds

Let us now discuss the deformation problem for complex structures on a complex manifold. As before, the idea will be to construct suitable versal deformations of a given complex structure J. Consider the space $\mathcal{J}(X) = \{J \in \mathrm{End}(TM) \big| J^2 = -\mathrm{Id}\}$ of almost complex structures on X. Note that the main difference with the case of holomorphic bundles $(E, \overline{\partial}_E)$, the space $\mathcal{J}(X)$ is not affine. Nonetheless, one can furnish it with the structure of an infinite-dimensional manifold.

Proposition 10.10. *The tangent space of $\mathcal{J}(X)$ is*

$$T\mathcal{J}(X) \coloneqq \mathcal{A}^{0,1}(\tau_X)$$

Proof. Let J_t be a smooth path of almost complex structures. This induces a continuous family of splittings $T^{1,0}_t X \oplus T^{0,1}_t X \cong T^*_{\mathbb{C}} X$. For small $t > 0$, we can consider the endomorphism $\phi \in \mathcal{A}_X\big(\mathrm{Hom}(T_0^{0,1}, T_t^{1,0})\big) \cong \mathcal{A}_X^{0,1}(T_t^{1,0})$, given by

$$\phi_t(v) = \mathrm{proj}_{T_t^{1,0}}(v)$$

for $v \in T_0^{0,1}$. This completely captures the difference between the J_t and J_0. Taking the limit $t \to 0$, we get $T_t^{1,0} \cong T_0^{1,0} (= \tau_X)$. □

Let $J_0 \in \mathcal{J}$ be a(n integrable) complex structure, and consider $J_\phi = (\mathrm{Id} + \phi) J_0 \in \mathcal{J}$ in a neighbourhood of J_0. We want to understand under what conditions J_ϕ is a(n integrable) complex structure. We have

Proposition 10.11. *Let $\phi \in \mathcal{A}^{0,1}(\tau_X)$ such that $\phi(T^{0,1}X) \cap T^{1,0}X = 0$, where $T^*_{\mathbb{C}}X \cong T^{1,0}X \oplus T^{0,1}X$ is the splitting induced by J_0. The condition of the almost complex structure J_ϕ being integrable is equivalent to*

$$\overline{\partial}\phi + \frac{1}{2}[\phi, \phi] = 0 \ . \tag{30}$$

Proof. Let $\widetilde{X}, \widetilde{Y} \in T_\phi^{0,1}$, so there exist $X, Y \in T^{0,1}$ such that $\widetilde{X} = X + \phi(X)$ and $\widetilde{Y} = Y + \phi(Y)$. The condition that $T_\phi^{0,1}$ is involutive is is equivalent $\phi\left([\widetilde{X}, \widetilde{Y}]^{0,1}\right) + [\widetilde{X}, \widetilde{Y}]^{1,0} \in \phi(T^{0,1}X) \cap T^{1,0}X$, where the $(0,1)$ and $(1,0)$ projections are taken with respect to the original complex structure. By assumption, this $\phi(T^{0,1}X) \cap T^{1,0}X = 0$, so we get the equation

$$\phi\left([\widetilde{X}, \widetilde{Y}]^{0,1}\right) + [\widetilde{X}, \widetilde{Y}]^{1,0} = 0 \tag{31}$$

Let us compute $[\widetilde{X}, \widetilde{Y}]$:

$$\begin{aligned} [\widetilde{X}, \widetilde{Y}] &= [X + \phi(X), Y + \phi(Y)] \\ &= [X, Y] + [X, \phi(Y)] - [Y, \phi(X)] + \frac{1}{2}[\phi, \phi](X, Y) \\ &= [X, Y] + \big(\overline{\partial}\phi(X, Y) - \phi([X, Y])\big) + \frac{1}{2}[\phi, \phi](X, Y) \ , \end{aligned}$$

where in the last step we used the vector-valued identity

$$d\phi(X,Y) = [X,\phi(Y)] - [Y,\phi(X)] - \phi([X,Y])$$

that generalises the usual formula for 2-forms (cf. Exercise 65).

Now, notice that, since J_0 is assumed to be involutive, the term $[X,Y]$ is purely of type $(0,1)$, while $[\widetilde{X},\widetilde{Y}] - [X,Y]$ is purely of type $(1,0)$. Therefore, Equation (31) becomes

$$\phi\left([X,Y]\right) + \left(\overline{\partial}\phi(X,Y) - \phi([X,Y])\right) + \frac{1}{2}[\phi,\phi](X,Y) = \left(\overline{\partial}\phi + \frac{1}{2}[\phi,\phi]\right)(X,Y) = 0\,. \qquad \square$$

If the reader is more comfortable with a computational proof using coordinates, we refer them to [Huy05, Lemma 6.1.2].

Note the difference between Equations (29) and (30). In the former case, the bracket is induced by the (pointwise) Lie algebra structure of the endomorphism bundle, whilst in the latter, it comes from the Lie bracket of vector fields.

As in the previous case, the action of the diffeomorphism group produces trivial solutions to the deformation problem, so we define the moduli space of complex structures on X to be the vanishing locus of

$$\begin{aligned} \Psi : \mathcal{J} &\to \mathcal{A}^{0,2}_X(\tau_X) \\ J &\mapsto N_J \end{aligned}$$

modulo the diffeomorphism group; $\mathcal{M}_X \coloneqq \Psi^{-1}(0)/\operatorname{Diff}(X)$.

Notice that this problem is a priori more complicated than the holomorphic bundle deformation problem, since the domain $\mathcal{J}$ is not modelled on a fixed vector space, and the codomain depends implicitly on the almost complex structure J.

Fortunately, this is not a problem from the point of view of the Banach Implicit Function Theorem 10.4. As in the case of holomorphic vector bundles, we consider the versal deformation $[J]$ given by the vanishing locus of

$$\begin{aligned} \widehat{\Psi} : \mathcal{A}^{0,1}_X(\tau_X) &\to \mathcal{A}^{0}_X(\tau_X) \oplus \mathcal{A}^{0,2}_X(\tau_X) \\ \phi &\mapsto \left(\overline{\partial}^*\phi, \overline{\partial}\phi + \frac{1}{2}[\phi,\phi]\right) \end{aligned}$$

Proceeding as before, one readily sees that its differential is $d\widehat{\Psi}(\phi) = (\overline{\partial}^*\phi, \overline{\partial}\phi)$. Again, from the Hodge decomposition, we have

$$\ker(d\Psi) = H^1(X,\tau_X) \qquad\qquad \operatorname{coker}(d\Psi) = H^0(\tau_X) \oplus H^2(\tau_X) \oplus \mathcal{A}^{0,3}_X(\tau_X)\,.$$

As in the case of holomorphic bundles, we see that the obstruction map takes values in $H^0(\tau_X) \oplus H^2(\tau_X)$, so we can apply the Banach Implicit Function Theorem 10.4, to get

Theorem 10.12 (Kuranishi)**.** *Let (X, J) be a compact complex manifold. Then there is a holomorphic map Θ from a neighbourhood of 0 in $H^1(X, \tau_X)$ to $H^2(X, \tau_X)$, such that*

(i) the family $S = \Theta^{-1}(0)$ is a versal deformation of the complex structure J,

(ii) the 2-jet of Θ is given by the bracket $[\cdot, \cdot] : H^1(X, \tau_X) \times H^1(X, \tau_X) \to H^2(X, \tau_X)$,

(iii) if $\mathrm{Aut}(X)$ is reductive, the map Θ can be chosen to be $\mathrm{Aut}(X)$-equivariant,

(iv) if $H^0(X, \tau_X) = 0$, the versal deformation S is universal, and a neighbourhood of $[J]$ in $\mathcal{M}_J$ is homeomorphic to the underlying space of the quotient of S by the discrete group $\mathrm{Aut}(X)$, and

(v) if $H^0(X, \tau_X) \neq 0$, a neighbourhood of $[J]$ in $\mathcal{M}_J$ is homeomorphic to the underlying space of the quotient $S/\mathrm{Aut}(X)$.

Let us make a few immediate remarks. By dualising the argument in Proposition 9.16, we get

Proposition 10.13. *Let (X, J) be a complex manifold such that the anti-canonical bundle K_X^* is positive. Then $H^2(\tau_X) = 0$.*

A complex manifold with positive anti-canonical bundle is called a *Fano* manifold.

It is worth noting that $H^1(X, \tau_X) \cong H^{n-1}(X, \Omega_X^1 \otimes K_X)^*$ lies precisely in the case $p + q = n$, where the Kodaira vanishing theorem gives no information regardless of the positivity/negativity of K_X.

Similarly, since on a weak Calabi–Yau manifold we have the isomorphism $\tau_X \cong \Omega_X^{n-1}$, we have

Proposition 10.14. *Let (X, J) be a weak Calabi–Yau manifold. Then*

$$\dim H^1(X, \tau_X) = \dim H^1(X, \Omega_X^{n-1}) \qquad \dim H^2(X, \tau_X) = \dim H^2(X, \Omega_X^{n-1}) \ .$$

As in the case of holomorphic vector bundles, we set up the *virtual dimension* of the moduli space to be

$$\text{v-dim}(\mathcal{M}_X) = H^1(X, \tau_X) - H^0(X, \tau_X) - H^2(X, \tau_X) \ .$$

Lemma 10.15. *The virtual dimension of a Riemann surface of genus g, Σ_g, is $3g - 3$.*

Proof. By the Hirzebruch–Riemann–Roch Theorem 7.24, we have

$$\text{v-dim}(\mathcal{M}_{\Sigma_g}) = -\int_{\Sigma_g} \mathrm{ch}(\Sigma_g)\,\mathrm{Td}(\Sigma_g) = -\frac{3}{2}\int_{\Sigma_g} c_1(\Sigma_g) = 3g - 3 \ . \qquad \square$$

Let us analyse the three cases $g = 0$, $g = 1$ and $g \geq 2$, which correspond to $c_1(K_X) < 0$, $c_1(K_X) = 0$ and $c_1(K_X) > 0$ respectively.

1. $\underline{\Sigma_0 \cong \mathbb{CP}^1}$: In this case, the virtual dimension is -3. From the Euler sequence (cf. Exercise 63), we have

$$h^0(\tau_{\mathbb{CP}^1}) = 3 \qquad h^1(\tau_{\mathbb{CP}^1}) = 0 \qquad h^2(\tau_{\mathbb{CP}^1}) = 0 \ .$$

In particular, the (connected component) of the moduli space is simply a point, with a trivial action of $\mathrm{PGL}(2, \mathbb{C})$ on it.

2. $\underline{\Sigma_1 \cong \mathbb{T}^1}$: In this case, the virtual dimension is 0. From Exercise 68, we have
$$h^0(\tau_{\mathbb{T}^1}) = 1 \qquad h^1(\tau_{\mathbb{T}^1}) = 1 \qquad h^2(\tau_{\mathbb{T}^1}) = 0 \ .$$
So, despite the virtual dimension of the moduli space being zero, the moduli space $\mathcal{M}_{\mathbb{T}}$ has complex dimension one.

3. $\underline{\Sigma_g \text{ for } g \geq 2}$: In this case, the virtual dimension is $3g - 3 > 0$. Moreover, by virtue of Proposition 9.16 and dimensional reasons, we have
$$h^0(\tau_{\Sigma_g}) = 0 \qquad h^1(\tau_{\Sigma_g}) = 3g - 3 \qquad h^2(\tau_{\Sigma_g}) = 0 \ .$$
In this case, the versal deformation space $\mathcal{T}_g$ is called the Teichmüller space of Σ_g, and the moduli space $\mathcal{M}_J$ is given by the quotient of $\mathcal{T}_g$ by the discrete group known as the mapping class group $MCG(\Sigma_g)$. Since it is a quotient by the mapping class group, the moduli space $\mathcal{M}_J$ is a complex orbifold, with quotient singularities at classes $[J]$ that have some finite group of symmetry.

Similarly, for complex surfaces, we have

Proposition 10.16. *Let S be a compact complex surface. Then*
$$\text{v-dim}(\mathcal{M}_S) = \frac{7}{2}\sigma(S) + \frac{3}{2}\chi(S) \ ,$$
where $\chi(S)$ is the (topological) Euler characteristic of S.

Proof. Once more, by the Hirzebruch–Riemann–Roch Theorem 7.24, we have
$$\begin{aligned}\text{v-dim}(\mathcal{M}_S) &= -\int_S \mathrm{ch}(S)\,\mathrm{Td}(S) \\ &= -\int_S \left(2 + c_1(S) + \frac{1}{2}\big(c_1^2(S) - 2c_2(S)\big)\right) \wedge \left(1 + \frac{c_1(S)}{2} + \frac{c_1^2(S) + c_2(S)}{12}\right) \\ &= \frac{5}{6}\langle c_2(S), [S]\rangle - \frac{7}{6}\langle c_1^2(S), [S]\rangle \ .\end{aligned}$$
Using that $\chi(S) = c_2(S)$ and $3\sigma(S) = c_1^2(S) - 2c_2(S)$, the claim follows. □

10.3 The Calabi–Yau case and the Tian-Todorov lemma

We saw that Fano manifolds are always unobstructed but can have large symmetry groups. Conversely, manifolds of general type have a discrete group of automorphisms, but their deformation will be generally obstructed.

That leaves us (besides the indefinite case) with the case of weak Calabi–Yau manifolds. The obstruction group is given by $h^{n-1,2}$, which does not generally vanish. For instance, in $n = 3$, we always have $h^{2,2} \neq 0$. In Exercise 68, the reader is invited to prove that for flat tori, the obstruction map vanishes, even though $h^{n-1,2} = \binom{n}{n-1}\binom{n}{2} = \frac{n^2(n-1)}{2}$. The goal of this final section will be to prove that

Theorem 10.17 (Bogomolov–Tian–Todorov)**.** *Let (X, J) be a Kähler weak Calabi–Yau manifold. The map*

$$\begin{aligned}[\cdot,\cdot] : H^1(X,\tau_X) \times H^1(X,\tau_X) &\to H^2(X,\tau_X) \\ (\phi,\psi) &\mapsto [\phi,\psi]\end{aligned}$$

is zero. In particular, the obstruction map $\Theta(\phi) \coloneqq [\phi, \phi]$ vanishes.

Note that the Kähler condition is vacuous if (X, J) is a strict weak Calabi–Yau of dimension ≥ 3 by Corollary 8.23.

To prove the theorem, let us understand the geometry of the weak Calabi–Yau case better. For the remainder of the section, (X^n, J, ω) will denote a closed Calabi–Yau manifold of complex dimension n. Since $K_X \cong \mathcal{O}_X$, we can choose a nowhere-zero holomorphic section Ω such that $\langle\Omega\rangle = H^0(X, K_X)$. We have an explicit isomorphism

$$\begin{aligned}\lrcorner : \mathcal{A}^0_X(\tau_X) &\to \mathcal{A}^{n-1,0}(X) \\ v &\mapsto v\lrcorner\Omega\ ,\end{aligned}$$

and by extension, $\lrcorner^q : \mathcal{A}^{0,q}_X(\tau_X) \to \mathcal{A}^{n-1,q}(X)$. This makes the isomorphisms $H^q(\tau_X) \cong \mathcal{H}^{n-1,q}$ explicit, which appeared in previous sections via Serre duality.

The key computation, due to Tian and Todorov, is the following:

Lemma 10.18 (Tian–Todorov)**.** *Let $\alpha, \beta \in \mathcal{A}^{0,1}(\tau_X)$. Then*

$$[\alpha,\beta]\lrcorner\Omega = \partial\big(\alpha\lrcorner(\beta\lrcorner\Omega)\big) - \langle\partial(\alpha\lrcorner\Omega),\Omega\rangle\ \beta\lrcorner\Omega + \langle\partial(\beta\lrcorner\Omega),\Omega\rangle\ \alpha\lrcorner\Omega\ .$$

Unfortunately, the only proof we are aware of for this result is using coordinates, at least if one does not further assume that the underlying manifold carries a Calabi–Yau metric. It would be interesting to produce a coordinate-free proof of the lemma.

Proof. The lemma is a local statement, so it suffices to prove it in local coordinates. Let $z_1, \ldots, z_n$ holomorphic coordinates, such that $\Omega = f dz_1 \wedge \cdots \wedge dz_n$ for $F \in \Omega^0_X$ a holomorphic function. Since the claim is linear (check it), it suffices to prove it for $\alpha = a\ d\overline{z_I} \otimes \frac{\partial}{\partial z_i}$ and $\beta = b\ d\overline{z_J} \otimes \frac{\partial}{\partial z_j}$, for $a, b \in \mathcal{A}^0_X$ functions. To lighten notation, we set $\widehat{dz_i} \coloneqq dz_1 \wedge \cdots \wedge dz_{i-1} \wedge dz_{i+1} \wedge \cdots \wedge dz_n$, so

$\alpha \lrcorner \Omega = fa\widehat{dz_i} \wedge d\overline{z_I}$. Plugging in the definitions, by the Leibniz rule, we have

$$\begin{aligned}
[\alpha, \beta] \lrcorner \Omega &= \left(\left[a\, d\overline{z_I} \otimes \frac{\partial}{\partial z_i}, b\, d\overline{z_J} \otimes \frac{\partial}{\partial z_j}\right]\right) \lrcorner \Omega = \left(a \frac{\partial b}{\partial z_i} \partial_{z_j} - b\, \frac{\partial a}{\partial z_j} \partial_{z_i}\right) \lrcorner \Omega \wedge (d\overline{z_I} \wedge d\overline{z_J}) \\
&= f \left(a \frac{\partial b}{\partial z_i} \widehat{dz_j} - b\, \frac{\partial a}{\partial z_j} \widehat{dz_i}\right) \wedge (d\overline{z_I} \wedge d\overline{z_J}) \\
&= \left[\left(\frac{\partial(abf)}{\partial z_i} \widehat{dz_j} - b \frac{\partial af}{\partial z_i} \widehat{dz_j}\right) - \left(\frac{\partial(abf)}{\partial z_j} \widehat{dz_i} - a \frac{\partial bf}{\partial z_j} \widehat{dz_i}\right)\right] \wedge (d\overline{z_I} \wedge d\overline{z_J}) \\
&= \left(\frac{\partial(abf)}{\partial z_i} \widehat{dz_j} - \frac{\partial(abf)}{\partial z_j} \widehat{dz_i}\right) \wedge (d\overline{z_I} \wedge d\overline{z_J}) - \langle \partial(\alpha \lrcorner \Omega), \Omega \rangle \beta \lrcorner \Omega + \langle \partial(\beta \lrcorner \Omega), \Omega \rangle \alpha \lrcorner \Omega \\
&= \partial \left(abf(\widehat{dz_j \wedge dz_i})\right) \wedge (d\overline{z_I} \wedge d\overline{z_J}) - \langle \partial(\alpha \lrcorner \Omega), \Omega \rangle \beta \lrcorner \Omega + \langle \partial(\beta \lrcorner \Omega), \Omega \rangle \alpha \lrcorner \Omega \\
&= \partial\big(\alpha \lrcorner (\beta \lrcorner \Omega)\big) - \langle \partial(\alpha \lrcorner \Omega), \Omega \rangle \beta \lrcorner \Omega + \langle \partial(\beta \lrcorner \Omega), \Omega \rangle \alpha \lrcorner \Omega\ . \qquad \square
\end{aligned}$$

With this lemma in hand, we give a proof of the Bogomolov–Tian–Todorov Theorem 10.17:

Proof of Theorem 10.17. Let $\phi, \psi \in H^1(X, \tau_X)$, and representatives $\alpha, \beta \in \mathcal{A}_X^{0,1}(\tau_X)$. The conditions $\overline{\partial}_{\tau_X} \alpha = 0 = \overline{\partial}_{\tau_X} \beta$ translate to $\overline{\partial}(\alpha \lrcorner \Omega) = 0 = \overline{\partial}(\beta \lrcorner \Omega)$ since Ω is holomorphic. We need to prove that $[\alpha, \beta]$ is $\overline{\partial}_{\tau_X}$-exact. Equivalently, that $[\alpha, \beta] \lrcorner \Omega$ is $\overline{\partial}$-exact.

Since X is assumed to be Kähler, $\overline{\partial}$-closedness is equivalent to ∂-closedness, so $\partial(\alpha \lrcorner \Omega) = 0 = \partial(\beta \lrcorner \Omega)$, and the Tian–Todorov Lemma implies $[\alpha, \beta] \lrcorner \Omega = \partial\big(\alpha \lrcorner (\beta \lrcorner \Omega)\big)$. Thus, $[\alpha, \beta] \lrcorner \Omega$ is ∂-exact and so it is also $\overline{\partial}$-exact, as needed. $\square$

We conclude the discussion by introducing the period map for weak Calabi–Yau manifolds:

Definition 10.19. Let (X, J) be a Kähler weak Calabi–Yau manifold. We define the *period map* of X as

$$\begin{aligned}
\Pi : \mathcal{M}_X &\to \mathbb{P}\big(H^m(X, \mathbb{C})\big) \\
[J] &\mapsto [\Omega_J]\ ,
\end{aligned}$$

where Ω_J is a nowhere vanishing section of the canonical bundle of X with the complex structure J.

By choosing a (fixed) local basis of $H_n(X, \mathbb{Z})$, $\{\gamma_1, \dots, \gamma_k\}$ we get a local description of the period map in terms of the periods of Ω_J with respect to the γ_i (hence the name!):

$$J \mapsto \left[\int_{\gamma_1} \Omega_J \ :\ \dots\ :\ \int_{\gamma_n} \Omega_J\right]\ .$$

Notice that the map is well-defined, since any other nowhere vanishing section is a multiple of Ω_J. Moreover, the tangent space of $\mathcal{M}_X$ is canonically isomorphic to $H^{1,n-1}(X, \mathbb{C}) \otimes H^{0,n}(X, \mathbb{C})^*$, which is a vector subspace of $T_{[\Omega]}\mathbb{P}\big(H^m(X, \mathbb{C})\big)$. In fact, one can prove

Theorem 10.20 (Griffiths)**.** *The period map is a holomorphic immersion.*

Thus, the period map provides a powerful bridge between the intrinsic deformation theory of complex structures and the extrinsic variation of Hodge structure encoded in cohomology.

References

[BPV84] W. Barth, C. Peters, and A. Van de Ven. *Compact Complex Surfaces*. Springer Verlag, 1984.

[CW21] J. Cirici and S. O. Wilson. "Dolbeault cohomology for almost complex manifolds". In: *Advances in Mathematics* 391 (2021). DOI: https://doi.org/10.1016/j.aim.2021.107970.

[Dem12] J.-P. Demailly. *Complex Analytic and Differential Geometry*. 2012.

[DK90] S. K. Donaldson and P. B. Kronheimer. *The geometry of four-manifolds*. Oxford mathematical monographs. 1990.

[GH80] A. Gray and L. Hervella. "The sixteen classes of almost Hermitian manifolds and their linear invariants". In: *Annali di Matematica Pura ed Applicata* 123 (1980), pp. 35–58. DOI: 10.1007/BF01796539.

[Har77] R. Hartshorne. *Algebraic Geometry*. Graduate texts in Mathematics - Springer, 1977.

[Huy05] D. Huybrechts. *Complex Geometry: An Introduction. Universitext*, Springer-Verlag, 2005.

[Joy20] D. Joyce. *Complex manifolds and Kähler geometry*. 2020. URL: https://people.maths.ox.ac.uk/joyce/KahlerGeom2022/index.html.

[Kob59] S. Kobayashi. "On the Automorphism Group of a Certain Class of Algebraic Manifolds". In: *Tohoku Mathematical Journal* 11 (2 1959), pp. 184–190.

[Kod86] K. Kodaira. *Complex Manifolds and Deformation of Complex Structures*. Springer Verlag, 1986.

[Lan83] S. Lang. *Real Analysis*. Addison–Wesley, Reading, Massachusetts, 1983.

[MS74] J. Milnor and J.D. Stasheff. *Characteristic Classes*. Princeton University Press, 1974.

[Mor07] A. Moroianu. *Lectures on Kähler Geometry*. London Mathematical Society Student Texts. Cambridge University Press, 2007.

[Tho06] R. P. Thomas. "Notes on GIT and symplectic reduction for bundles and varieties". In: *Surveys in Differential Geometry* 10 - A Tribute to Professor S.-S. Chern (2006).

[Wel08] R. O. Wells. *Differential Analysis on Complex Manifolds*. Graduate texts in Mathematics - Springer, 2008.

Exercise compilation

Exercises Week 1

Exercise 3. *Prove the following theorems for holomorphic functions in n variables $f : U \to \mathbb{C}$ with $U \subseteq \mathbb{C}^n$:*

(i) Open mapping theorem

(ii) Maximum principle

(iii) (Generalised) Liouville theorem

Exercise 4. *Prove that there is a one-to-one correspondence between holomorphic germs of $\mathcal{O}_{\mathbb{C}^n,0}$ and convergent power series in $\mathbb{C}[[z_1, \ldots, z_n]]$.*

Exercise 5. *Prove the n-dimensional Cauchy Integral Formula in detail:*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial D_R(z)} \frac{f(w_1, \ldots, w_n)}{(w_1 - z_1) \ldots (w_n - z_n)} dw_1 \ldots dw_n \ .$$

Exercise 6. *Finish the proof of the Weierstrass Division Theorem.*

Exercise 7. *Verify the following statements about analytic and holomorphic germs:*

(i) For any subset $A \subseteq \mathcal{O}_{X,x}$, $Z(A)$ is a well-defined analytic germ with $Z(A) = Z((A)_{\mathcal{O}_{X,x}})$.

(ii) For every analytic germ Z, $I(Z) = \{f \in \mathcal{O}_{X,x} \mid Z \subset Z(f)\}$ is an ideal.

(iii) If $X_1 \subset X_2$ are analytic germ, then $I(X_2) \subset I(X_1)$.

(iv) If $I_1 \subset I_2$ are ideals in $\mathcal{O}_{X,x}$, then $Z(I_2) \subset Z(I_1)$.

(v) $Z = Z(I(Z))$ and $I \subset I(Z(I))$.

(vi) $Z(I \cdot J) = Z(I) \cup Z(J)$ and $Z(I + J) = Z(I) \cap Z(J)$.

Exercise 8. *Prove the weak Nullstellensatz: If $f \in \mathcal{O}_{X,x}$ is irreducible and $g \in I(Z(f))$, then $f \mid g$.*

Exercise 9. *Prove that there is a one-to-one correspondence between convergent power series and holomorphic germs.*

Exercises Week 2

Exercise 10. *Let (M, J) be an almost complex manifold.*

(i) Show that $\dim_{\mathbb{R}} M$ is even and M is naturally oriented.

(ii) Show that there is at most one structure of a complex manifold inducing J.

Exercise 11. *Show that the only compact connected complex submanifold in $\mathbb{C}^n$ is a point.*

Exercise 12. *Show that any Hopf surface*

$$H^2_\lambda := \mathbb{C}^2 \setminus \{(0,0)\}/(z_1, z_2) \sim \lambda(z_1, z_2), \quad \lambda \in (0,1)$$

contains many elliptic curves.

Exercise 13. *The usual definition of the Nijenhuis is*

$$\widetilde{N_J}(X,Y) = [X,Y] + J([JX,Y] + [X,JY]) - [JX,JY] \, .$$

Prove that the two definitions are equivalent (up to complexification and conjugation).

Exercise 14. *Let $f : \mathbb{CP}^n \to \mathbb{C}^m/\Lambda$ be a holomorphic map. Then f is constant.*

*(**Hint**: think about how f interacts with the covering maps of the tours).*

Exercise 15. *Let (M, J) be an almost complex manifold and consider the associated ∂ and μ operators. Prove that they satisfy the following properties:*

(i) The Leibniz rule.

(ii) ∂ is $\mathbb{C}$-linear and μ is function linear.

(iii) The following identities hold:

$$\begin{aligned} &\mu\partial + \partial\mu = 0 \, , \qquad \partial^2 + \overline{\partial}\mu + \mu\overline{\partial} = 0 \, , \\ &\mu^2 = 0 \, , \qquad \mu\overline{\mu} + \overline{\partial}\partial + \partial\overline{\partial} + \overline{\mu}\mu = 0 \, . \end{aligned}$$

Exercise 16 (Serge and Plücker embeddings)**.**

(i) Show that there is a holomorphic embedding $\mathbb{CP}^n \times \mathbb{CP}^m \to \mathbb{CP}^{(n+1)(m+1)-1}$

(ii) Write down the corresponding homogeneous equations in the case $n = m = 1$.

(iii) Show that the map

$$\begin{array}{rcl} Gr(k,n) & \to & \mathbb{P}(\Lambda^k \mathbb{C}^n) \\ W = \operatorname{span}\{w_1, \ldots, w_k\} & \mapsto & [w_1 \wedge \cdots \wedge w_k] \end{array}$$

is a holomorphic embedding.

(iv) Write down the corresponding homogeneous equations for the Grassmannian $Gr(2,4)$.

Exercise 17. *Let $B = B_r(0) \subseteq \mathbb{C}^n$ be a polydisc. Then, the cohomology groups $H^{p,q}_{BC}(B)$ and $H^{p,q}_A(B)$ vanish for $p, q > 0$.*

Exercises Week 3

Exercise 18. *Let X be a topological space, and A be an abelian group. Consider the presheaves*

- *$\mathcal{F}_1(U) = A$ for all open sets $U \subseteq X$,*

- $\mathcal{F}_2(U) = A$ *for all open sets* $\emptyset \neq U \subseteq X$ *and* $\mathcal{F}_2(\emptyset) = 0$,

with the obvious restriction maps in all cases.

(i) Show that neither $\mathcal{F}_1$ *or* $\mathcal{F}_2$ *are sheaves.*

(ii) Show that they have the same étale space, given

$$\acute{E}t(\mathcal{F}^{const}) = X \times A$$

where A *is endowed with the discrete topology, and so* $\mathcal{F}_i^+ = \underline{A}$.

(iii) Compute $\underline{A}(U)$ *for a given open set* U.

Exercise 19. *Show that a sequence of sheaves*

$$\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$$

is exact if and only if the corresponding sequence of stalks

$$\mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_\S \xrightarrow{\beta_x} \mathcal{H}_x$$

is exact for all $x \in X$.

Exercise 20. *Show that for a topological space* $X, U \subset X$, *a closed subset with inclusion map* $j : U \hookrightarrow X$ *and a sheaf* $\mathcal{F}$ *of abelian groups on* U,

$$H^q(U, \mathcal{F}) \cong H^q(X, j_*\mathcal{F})$$

for all $q > 0$.

Exercise 21. *Let* X *be a complex manifold. We say a sheaf is a* coherent sheaf *if every point* $x \in X$ *has an open neighbourhood* U *in* X *an an exact sequence*

$$\mathcal{O}_X^p|_U \to \mathcal{O}_X^q|_U \to \mathcal{F}|_U \to 0$$

for some natural numbers p *and* q, *with* $\mathcal{O}_X^k = \bigoplus_{i=1}^k \mathcal{O}_X$.

(i) Show that the sheaf of sections of a holomorphic vector bundle is a coherent sheaf.

(ii) Show that a coherent sheaf where we can take $p = 0$ *for all points has an associated holomorphic bundle associated to it.*

Let $\iota : Z \hookrightarrow X$ *be a closed submanifold in* X.

(iii) Show that the ideal sheaf of Z, *denoted by* $\mathcal{I}_Z$, *is coherent, where*

$$\mathcal{I}_Z(U) = \Big\{ f \in \mathcal{O}_X(U) \Big| \, f|_Z = 0 \Big\},$$

for $U \subseteq X$ *open.*

(iv) Show that the direct image $\iota^*\mathcal{O}_Z$ *is a coherent sheaf.*

(v) Prove that there is a short exact sequence of sheaves:

$$0 \to \mathcal{I}_Z \to \mathcal{O}_X \to \iota^* \mathcal{O}_Z \to 0 \ .$$

Exercise 22. *Show that*

$$H^q(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \cong \begin{cases} 0 & \text{for } q \neq 0 \\ \mathbb{C} & \text{for } q = 0 \end{cases} .$$

Exercises Week 4

Exercise 23. *Prove that, up to vector bundle isomorphism, we have the following correspondences:*

- *real vector bundles of rank r* $\xleftrightarrow{1:1} \check{H}^1\big(X, \mathrm{GL}(r, \mathcal{C}^\infty(X, \mathbb{R})\big)$,
- *complex vector bundles of rank r* $\xleftrightarrow{1:1} \check{H}^1\big(X, \mathrm{GL}(r, \mathcal{C}^\infty(X, \mathbb{C})\big)$
- *holomorphic vector bundles of rank r* $\xleftrightarrow{1:1} \check{H}^1\big(X, \mathrm{GL}(r, \mathcal{O}_X)\big)$,

where $\mathrm{GL}(r, \mathcal{F})$ *is the sheaf of invertible rank k matrices with coefficients in the sheaf $\mathcal{F}$.*

Exercise 24. *Show that any short exact sequence of holomorphic vector bundles*

$$0 \longrightarrow L \longrightarrow E \longrightarrow F \longrightarrow 0 \ ,$$

where L is a line bundle, induces the short exact sequence

$$0 \longrightarrow L \otimes \Lambda^{k-1} E \longrightarrow \Lambda^k E \longrightarrow \Lambda^k F \longrightarrow 0.$$

Exercise 25. *Let L_1 and L_2 be holomorphic line bundles such that $L_1|_{X \setminus Y} = L_2|_{X \setminus Y}$ for $Y \subset X$ a complex submanifold of codimension ≥ 2. Show that $L_1 \cong L_2$.*

Exercise 26. *Prove that for a 1-dimensional connected complex manifold X,*

$$K(X) \cong \big\{ f : X \to \mathbb{P}^1 \big| \ f \text{ is holomorphic, } f \not\equiv \infty \big\} \ .$$

Exercise 27. *Let $X = \mathbb{C}^n / \Lambda$ be a complex torus. Show that:*

(a) $\tau_X = \mathcal{O}_X^n$, $K_X = \mathcal{O}_X$

(b) kod$(X) = 0$ *and kod*$(Y) \geq 0$ *for every (positive-dimensional) submanifold $Y \subset X$*

Exercise 28 (Veronese embedding)**.** *Show that the map defined by global sections $H^0(\mathbb{P}^1, \mathcal{O}(2))$ is an embedding*

$$\phi : \mathbb{P}^1 \hookrightarrow \mathbb{P}^2$$

realising $\mathbb{P}^1$ as the zero set of a homogeneous polynomial of degree 2.

Exercise 29. *Assume X is a connected complex manifold.*

(i) Show that a holomorphic line bundle L over X is isomorphic to the trivial bundle if and only if both L and L^* admit a non-trivial global section.

(ii) Deduce $H^0(\mathbb{P}^n, \mathcal{O}(k)) = 0$ for $k < 0$.

(iii) Prove (or convince yourself) that there exist line bundles such that

$$H^0(X, L) = 0 = H^0(X, L^*) \ .$$

Exercises Week 5

Exercise 30. *Prove the following hypersurfaces are smooth and compute their Kodaira dimension:*

(i) $Z(x_0^2 + z_1^2 + z_2^2) \subset \mathbb{CP}^2$

(ii) $Z(x_0^3 + z_1^3 + z_2^3) \subset \mathbb{CP}^2$

(iii) $Z(x_0^5 + \cdots + z_5^5) \subset \mathbb{CP}^4$

Exercise 31. *Show that the image of* $\mathrm{Div}(X) \to \mathrm{Pic}(X)$ *is given by the classes of line bundles which admit a non-zero meromorphic section.*

Exercise 32 (Bezout's theorem)**.** *Let $C, D \subset \mathbb{P}^2$ smooth (distinct) curves defined by homogeneous polynomials f and g of degrees d and e respectively.*

(i) Show that the line bundle $\mathcal{O}(1)$ restricted to C is of degree d.

(ii) Show that

$$d \cdot e = \sum_{p \in C \cap D} \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^2, g}/(f, g).$$

Exercise 33. *Let X be a smooth complex manifold with $K_X \cong \mathcal{O}_X$. Show that X cannot be obtained by a blow-up of another surface.*

Exercise 34. *Let X be a smooth Calabi–Yau manifold (with $K_X \cong \mathcal{O}_X$). Show that X cannot be obtained by a blow-up of another surface.*

Exercises Week 6

Exercise 35. *For all $n > 0$ and $k \in \mathbb{Z}$, compute*

$$H^n\big(\mathbb{CP}^n, \mathcal{O}(k)\big) \ .$$

Exercise 36. *Let $\big(V^{2n}, \langle \cdot\,,\,\cdot \rangle\big)$ be a Euclidean vector space.*

(i) Show that the space of compatible complex structure is parametrised by two disjoint copies of $\mathrm{SO}(2n)/\mathrm{U}(n)$.

(ii) Show that for $n = 2$, this corresponds to two copies of $\mathbb{CP}^1$.

(iii) Show that for $n = 3$, this corresponds to two copies of $\mathbb{CP}^3$.

*(**Hint:** Recall the exceptional isomorphisms $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$ and $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$.)*

For the remainder of the sheet, let $\big(V^{2n}, \langle \cdot , \cdot \rangle, I\big)$ denote a Euclidean vector space of real dimension $2n$ with compatible complex structure I.

Exercise 37. *Let $p, q, p', q' \in \mathbb{N}$ with $p + q = k = p' + q'$ and $k \leq n$.*

Consider the Hodge-Riemann pairing:

$$Q : \bigwedge^{p,q} V_{\mathbb{C}}^* \times \bigwedge^{p',q'} V_{\mathbb{C}}^* \to \mathbb{C}$$
$$(\alpha, \beta) \mapsto (-1)^{\binom{k}{2}} \alpha \wedge \beta \wedge \omega^{n-k} \; .$$

Show

(i) Q vanishes unless $(p, q) = (q', p')$.

(ii) For $0 \neq \alpha \in P^{p,q} \subseteq \Lambda^{p,q} V_{\mathbb{C}}^*$, we have

$$i^{p-q} Q(\alpha, \overline{\alpha}) = [n - (p + q)]! \, \langle \alpha, \alpha \rangle > 0 \; .$$

Exercise 38. *Let $x_1, y_1 = I(x_1), \ldots, x_n, y_n = I(x_n)$ be an orthonormal basis of V. Show that, for any $\alpha \in \Lambda^k V$*

$$\Lambda\alpha(X_1, \ldots, X_{k-2}) = \sum_{i=1}^{n} \alpha(x_i, y_i, X_1, \ldots, X_{k-2}) \; .$$

Exercise 39. *Let $(E, \overline{\partial}_E, h)$ be a holomorphic hermitian vector bundle on X. Show that*

(i) the space of connections on E is an affine space modelled on $\mathcal{A}^1_X\big(\mathrm{End}(E)\big)$;

(ii) the space of metric connections on E is an affine space modelled on $\mathcal{A}^1_X\big(\mathrm{End}(E, h)\big)$;

(iii) the space of compatible connections on E is an affine space modelled on $\mathcal{A}^{1,0}_X\big(\mathrm{End}(E)\big)$.

Use the above to give an alternate proof of the uniqueness of the Chern connection.

Exercise 40. *Prove that*

(i) $\overline{\partial}^* = - * \partial *$ and $\partial^* = - * \overline{\partial} *$,

(ii) $\Delta_{\overline{\partial}} = \overline{\Delta_{\partial}}$.

(iii) $\mathcal{H}^{p,q}_{\overline{\partial}} = \ker \overline{\partial} \; \cap \ker \overline{\partial}^*$.

Exercise 41. *The goal of this exercise is to establish a Hodge isomorphism for vector bundles. Let $(E, \overline{\partial}_E, h)$ be a holomorphic hermitian vector bundle.*

(i) Show that the Hodge star extends naturally to E-valued forms.

(ii) Show that $\overline{\partial}^*_E = - \overline{*} \overline{\partial}_E \overline{*}$ is an L^2-adjoint to $\overline{\partial}_E$ with respect to the L^2-inner product induced by h.

(iii) Assuming we have an analogous Hodge decomposition, reproduce the argument in the lectures to conclude there is an isomorphism

$$H^q(X, \Omega^p \otimes E) \cong \mathcal{H}^{p,q}(X, E) ,$$

where $\mathcal{H}^{p,q}(X, E) := \ker \Delta_{\overline{\partial}_E}$.

Exercises Week 7

Exercise 42. *Let* (X, g, J) *be a Kähler manifold and* $\iota : Y \hookrightarrow X$ *a closed complex submanifold. Use the Wirtinger inequality to prove that* Y *is volume-minimising within its homology class.*

Exercise 43. *Consider the open unit disk* $\mathbb{D} = \{z \in \mathbb{C} \big| \, |z| < 1\}$*. Prove that the* $(1,1)$*-form*

$$\omega = -\frac{i}{2}\partial\overline{\partial}\log\left(1 - |z|^2\right)$$

defines a Kähler metric on $\mathbb{D}$*.*

Exercise 44. *The goal of the exercise is to prove that the Iwasawa manifold* $\mathbb{I} = \mathbb{U}/\mathbb{U}_{\mathbb{Z}}$ *is not Kähler.*

(i) Prove that holomorphic forms on a Kähler manifold are harmonic.

(ii) Compute a basis of left-invariant holomorphic 1-forms for $\mathbb{I}$*.*

(iii) Conclude $\mathbb{I}$ *is a weak Calabi–Yau manifold.*

(iv) Deduce it does not admit a compatible Kähler metric.

Exercise 45. *Consider the hermitian metric*

$$h_z(s, s') = \frac{\langle s, s' \rangle}{\sum_i |z_i|^2}$$

on $\mathcal{O}(1)$*, induced by its global sections. Denote by* ∇ *its Chern connection. Prove*

(i) $\frac{i}{2\pi}F_\nabla = \omega_{FS}$*, where* ω_{FS} *is the Kähler for of the Fubini-Study metric.*

(ii) $\displaystyle\int_{\mathbb{CP}^n} \omega_{FS}^n = 1$

Exercise 46. *Show that, in dimension* ≥ 2*, there exists at most one Kähler metric in a given conformal class (up to scale).*

Exercise 47. *Let* $\operatorname{Pic}^0(X) := \ker\left(\operatorname{Pic}(X) \to H^2(X, \mathbb{Z})\right)$*.*

(i) Show that if X *is Kähler is a complex torus of dimension* $b^1(X)$*.*

(ii) Give a counterexample when X *is not Kähler.*

Exercises Week 8

Exercise 48.

(i) *Compute the signature of a closed complex surface S with $\kappa(S) = -\infty$.*

(ii) *Prove that $\widetilde{S}_{2n} = \#_{2n}\mathbb{CP}^2$ cannot be Kähler for $n \geq 2$.*

Exercise 49. *Consider the complex torus $\mathbb{T}_\Gamma = \mathbb{C}^2/\Gamma$ for a lattice Γ.*

(i) *Find Γ such that the only holomorphic line bundle in $\mathbb{T}_\Gamma$ is the trivial one.*

(ii) *Prove (or convince yourself) that this is in fact true for a generic Γ.*

(iii) *Conclude that most tori in dimensions ≥ 2 are not projective.*

(iv) *Compute the Picard group $\mathrm{Pic}(\mathbb{T}_\Gamma)$ for any lattice Γ.*

Exercise 50. *Let X be a complex manifold such that $h^{2,0}(X) = 0$. Show that X is projective.*

Conclude that Calabi–Yau manifolds (in the strict sense) of dimension ≥ 3 are all projective.

Exercise 51. *Let S be a strict weak Calabi–Yau surface.*

(i) *Prove that $\langle c_2(S), [S] \rangle = 24$.*

(ii) *Compute the signature of S.*

(iii) *Calculate the Hodge diamond of S.*

Exercise 52. *Let X be a Kähler manifold, with $\dim(X) \geq 4$, and consider $\iota : Y \to X$ a hypersurface with $\mathcal{O}(Y)$ positive. Assuming $H^2(X, \mathbb{Z})$ and $H^2(Y, \mathbb{Z})$ are torsion-free, prove that $\iota^* : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is an isomorphism.*

Exercise 53. *Let Σ be a Riemann surface. Prove that*

$$c_1(K_\Sigma) = 2g(\Sigma) - 2 \, .$$

*(**Recall:** For a conformal metric g, the Gaussian curvature is $K = \Delta \log(g)$.)*

Exercise 54. *Let Σ_d be a smooth hypersurface in $\mathbb{CP}^2$. Show that*

$$g(\Sigma_d) = \frac{(d-1)(d-2)}{2} \, .$$

Exercises Week 9

Exercise 55. *Let X, Y be closed complex manifolds. Prove that they are projective if and only their product is. Give two alternative proofs, using*

(i) *the Serge embedding, and*

(ii) *the Kodaira embedding theorem.*

Exercise 56. *Let $L \to \Sigma$ be a line bundle on a Riemann surface Σ. Prove the following assertions.*

(i) *If* $\deg(L) < 0$*, then* $H^0(\Sigma, L) = 0$.

(ii) *If* $\deg(L) > \deg(K_\Sigma)$*,* $H^1(\Sigma, L) = 0$.

(iii) *If* $\deg(L) > \deg(K_\Sigma) + 2$*, its pluricanonical map* ϕ_L *is an embedding.*

In particular, conclude that all Riemann surfaces are projective.

(iv) *Use the Hirzebruch–Riemann–Roch Theorem to compute* $\dim H^0(\Sigma, L)$ *when* $\deg(L) > \deg(K_\Sigma) + 1$.

Exercise 57. *Prove that the blow-up of a projective manifold is again projective.*

Exercise 58. *Show that any vector bundle E on a projective manifold X can be written as a quotient $(L^k)^{\oplus l} \to E$ with L an ample line bundle, $k \ll 0$ and $l \gg 0$.*

Exercise 59. *Prove that a complex torus $\mathbb{T}_\Gamma = \mathbb{C}^n/\Gamma$ is projective if and only if there exists an alternating bilinear form*

$$\omega : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$$

such that

- $\omega(iu, iv) = \omega(u, v)$,
- $\omega(\cdot, i\,\cdot)$ *is positive definite, and*
- $\omega(u, v) \in \mathbb{Z}$ *for all* $u, v \in \Gamma$

Exercise 60. *Let X be a compact Kähler manifold and*

$$Alb(X) := H^0(X, \Omega^1_X)^*/H_1(X, \mathbb{Z})$$

where

$$H_1(X, \mathbb{Z}) \to H^0(X, \Omega^1_X)^*, \quad \gamma \mapsto \left(\alpha \mapsto \int_\gamma \alpha\right).$$

Prove the following statements.

(i) $Alb(X)$ *is a complex torus.*

(ii) *Fixing a base point* $z_0 \in X$*, yields a holomorphic map* $X \to Alb(X)$.

(iii) *For* $\mathbb{T}_\Gamma = \mathbb{C}^n/\Gamma$*, the map* $\mathbb{T}_\Gamma \to Alb(\mathbb{T}_\Gamma)$ *is a biholomorphism.*

Exercises Week 10

Exercise 61. *Let (X, J) be a complex manifold. Show that $H^0(X, \tau_X)$ is naturally a complex Lie algebra, for which the Lie bracket is complex bilinear.*

*(**Hint:** Recall the Nijenhuis tensor $N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]$.)*

Exercise 62. *Prove the following assertions.*

(i) The group $\mathrm{Aut}(\mathbb{CP}^n)$ *is path-connected.*

(ii) Any $f \in \mathrm{Aut}(\mathbb{CP}^n)$*, with* $f \neq 1$*, has* $n+1$ *fixed points (counted with multiplicity).*

(iii) The group $\mathrm{Aut}(\mathbb{CP}^1)$ *is sharply 3-transitive.*

(iv) The group $\mathrm{Aut}(\mathbb{CP}^n)$ *is 2-transitive but not 3-transitive for* $n \geq 2$*.*

Exercise 63.

(i) Consider (E_1, h_1) *and* (E_2, h_2) *holomorphic hermitian vector bundles over a Kähler manifold. Prove that*

$$\mathcal{H}^{p,q}(E_1 \oplus E_2) \cong \mathcal{H}^{p,q}(E_1) \oplus \mathcal{H}^{p,q}(E_2) \ .$$

(ii) Using the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{CP}^n} \to \bigoplus_{i=0}^{n} \mathcal{O}(1) \to \tau_{\mathbb{CP}^n} \to 0 \ ,$$

show that

$$\dim H^i(\mathbb{CP}^n, \tau_{\mathbb{CP}^n}) = \begin{cases} (n+1)^2 - 1 & \text{if } i = 0 \ , \\ 0 & \text{otherwise} \end{cases} \ .$$

Exercise 64. *Let* $f : X \to X$ *be a continuous map. We say* z *is fixed by* f *if* $f(z) = z$*. For an isolated fix point, we define the index of* z *as the degree of the homology map induced in* $U \setminus \{z\}$ *for* U *a small enough neighbourhood of* z*.*

Assume that f *is holomorphic.*

(i) Show that the degree is always positive.

(ii) Using local coordinates, show that the degree is equal to the multiplicity of the zero of the (locally defined) function $g(z) = f(z) - z$*.*

Assume that X *is closed.*

(iii) Prove that if $f \neq Id$*, the number of fixed points (with multiplicity) coincides with the Lefschetz number of* f*:*

$$\Lambda_f = \sum_{k \geq 0} \mathrm{Tr}\left(f^* : H^k(X, \mathbb{R}) \to H^k(X, \mathbb{R})\right) \ .$$

(iv) Compute the number of fixed points if f *is homotopic to the identity.*

(v) Study the fixed points of $f \in \mathrm{Aut}(\mathbb{T}^{\mathrm{n}})$*.*

Exercises Week 11

Exercise 65. *Prove that the standard identity*

$$d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])$$

extends to vector-valued 1-forms ϕ*:*

$$d\phi(X,Y) = [X,\phi(Y)] - [Y,\phi(X)] - \phi([X,Y]) \ .$$

Exercise 66. *Consider* $H_\lambda = \mathbb{C}^n \setminus \{0\}/\langle\lambda\rangle$ *the Hopf surface, with* $\lambda \in \mathbb{C}^*$, $|\lambda| < 1$, *for* $n \geq 2$.

(i) Show that $f \in \mathrm{Aut}(H_\lambda)$ *extends to a biholomorphism* $\widetilde{f} : \mathbb{C}^n \to \mathbb{C}^n$ *with* $\widetilde{f}(0) = 0$.

(ii) Show that $\widetilde{f}(\lambda z) = \lambda \widetilde{f}(z)$ *(as opposed to* $\lambda \widetilde{f}(\lambda z) = \widetilde{f}(z)$*).*

(iii) Prove that $\partial \widetilde{f}(\lambda^n z) = \partial \widetilde{f}(z)$ *for all* $n \in \mathbb{Z}$, *and conclude that* $\widetilde{f}(z) = Az$ *for* $A \in \mathrm{GL}(n,\mathbb{C})$.

(iv) Compute $\mathrm{Aut}(H_\lambda)$.

Exercise 67. *Let* X *be a Kähler manifold with* $c_1(K_X) < 0$. *Prove that* $H^2(X,\tau_X) = 0$.

Exercise 68. *Consider a flat torus* $\mathbb{T}^n = \mathbb{C}^n/\Gamma$.

(i) Find an explicit basis of $\mathcal{H}^{p,q}(\mathbb{T}^n)$. *Show that* $h^{p,q} = \binom{n}{p}\binom{n}{q}$

(ii) Compute the virtual dimension of complex strucutres $\mathcal{M}_{\mathbb{T}^n}$.

(iii) Show that the obstruction map $H^1(\mathbb{T}^n,\tau_{\mathbb{T}^n}) \xrightarrow{\Phi} H^2(\mathbb{T}^n,\tau_{\mathbb{T}^n})$ *vanishes.*

(iv) Study how the group $\mathrm{Aut}^0(\mathbb{T}^n)$ *acts on* $H^1(\mathbb{T}^n,\tau_{\mathbb{T}^n})$.

(v) Compute the dimension of $\mathcal{M}^0_{\mathbb{T}^n} := \Psi^{-1}(0)/\mathrm{Aut}^0(\mathbb{T}^n)$.

Exercise 69 (Moduli space of complex tori)**.**

(i) Prove that a complex torus of dimension n *is determined by a map* $\mathbb{Z}^{2n} \to \mathbb{C}^n \cong \mathbb{R}^{2n}$, *and thus, can be identified with an element of* $\mathrm{GL}(2n,\mathbb{R})$.

So there is a surjective map $\mathrm{GL}(2n,\mathbb{R}) \to \mathcal{M}_{\mathbb{T}^n}$.

(ii) Prove the actions of $\mathrm{GL}(n,\mathbb{C})$ *and* $\mathrm{GL}(2n,\mathbb{Z})$ *on* $\mathrm{GL}(2n,\mathbb{R})$ *induce the trivial action on* $\mathcal{M}_{\mathbb{T}^n}$.

(iii) Conclude that $\mathrm{GL}(n,\mathbb{C}) \setminus \mathrm{GL}(2n,\mathbb{R})$ *is a "covering" space of* $\mathcal{M}_{\mathbb{T}^n}$.

(iv) Prove that $\mathcal{M}_{\mathbb{T}^n}$ *is isomorphic to the bi-quotient*

$$\mathrm{GL}(n,\mathbb{C}) \setminus \mathrm{GL}(2n,\mathbb{R})/\mathrm{GL}(2n,\mathbb{Z}) \ .$$

(v) Prove that $\mathrm{GL}(2n,\mathbb{Z})$ *does not act properly discontinuous on* $\mathrm{GL}(n,\mathbb{C}) \setminus \mathrm{GL}(2n,\mathbb{R})$ *for* $n \geq 2$, *so* $\mathcal{M}_{\mathbb{T}^n}$ *is not Hausdorff.*

(vi) For $n = 1$, *prove that*

$$\mathrm{GL}(1,\mathbb{C}) \setminus \mathrm{GL}(2,\mathbb{R}) \cong \mathbb{H}^2 \sqcup \mathbb{H}^2 \ ,$$

Where $\mathbb{H}^2 = \{x+iy \in \mathbb{C} \big| \ y > 0\}$ *is the hyperbolic plane, and the two copies are distinguished by orientation.*

(vii) Give a geometric description of $\mathcal{M}_{\mathbb{T}^1}$.

(viii) Show that $\mathcal{M}_{\mathbb{T}^1}$ is not compact. Can you find a natural compactification?

(ix) Show that $\mathcal{M}_{\mathbb{T}^1}$ is not a manifold, but an orbifold. Can you compute the stabiliser of the non-smooth points?

Exercise 70 (Moduli space of principally polarised tori). *From Exercise 59, we know that a complex torus $\mathbb{C}^n/\Gamma$ is projective iff ω restricts to a symplectic form $\Gamma \times \Gamma \to \mathbb{Z}$.*

Fix a symplectic structure ω (equivalently a hermitian metric) on $\mathbb{C}^n$.

(i) Prove that a projective torus of dimension n is determined by a symplectic map $\mathbb{Z}^{2n} \to \mathbb{C}^n \cong \mathbb{R}^{2n}$, and thus, can be identified with an element of $\mathrm{Sp}(2n, \mathbb{R})$.

We define $\mathcal{M}^n_{Ab}$ the moduli of principally polarised n-dimensional tori as the collection of tori given by the construction in (i), modulo automorphism.

(ii) Show the actions of $U(n)$ and $\mathrm{Sp}(2n, \mathbb{Z})$ on $\mathrm{Sp}(2n, \mathbb{R})$ induce the trivial action on $\mathcal{M}^n_{Ab}$.

(iii) Prove that $\mathcal{M}^n_{Ab}$ is isomorphic to the bi-quotient

$$\mathrm{U}(n) \setminus \mathrm{Sp}(2n, \mathbb{R}) / \mathrm{Sp}(2n, \mathbb{Z}) \ .$$

(iv) Check that, for $n = 1$, we have $\mathcal{M}^1_{Ab} \cong \mathcal{M}_{\mathbb{T}^1}$.

(v) Prove that $\mathrm{Sp}(2n, \mathbb{Z})$ acts properly discontinuously on $\mathrm{U}(n) \setminus \mathrm{Sp}(2n, \mathbb{R})$. In particular, the moduli space $\mathcal{M}^n_{Ab}$ is Hausdorff for $n \geq 2$.