

Lecture Notes on Complex Geometry

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This course aims to give an introduction to the world of complex geometry. The main idea I would like to convey to the reader is the strong local-to-global properties that holomorphic functions possess, and thus manifolds whose transition functions are holomorphic: complex manifolds.

I have based these notes on the two excellent books, the first by Daniel Huybrechts [Huy05] and the other by Jean-Pierre Demailly, [Dem12], who unfortunately passed away before the book was ever published, and only online drafts are available.

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1 Holomorphic functions: Local theory

We begin by reviewing fundamental properties of holomorphic functions and their generalisation to several complex variables. We identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$ as real vector spaces via the map $(z_1, \dots, z_n) \mapsto (x_1, y_1, \dots, x_n, y_n)$ where $z_j = x_j + iy_j$.

Definition 1.1. A function $f : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2m}$ is *differentiable at z_0* if there exists a linear map Df_{z_0} such that

$$f(z) = f(z_0) + Df_{z_0}(z - z_0) + o(\|z - z_0\|).$$

Definition 1.2. A function $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$ is *holomorphic at z_0* if it is real-differentiable and its differential Df_{z_0} is complex-linear, i.e., $Df_{z_0} \in \text{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^m) \cong \text{GL}(n, \mathbb{C}) \subseteq \text{GL}(2n, \mathbb{R})$.

The complex-linearity condition can be expressed using the standard complex structure J on $\mathbb{R}^{2n}$ (multiplication by i):

$$J \circ Df_{z_0} = Df_{z_0} \circ J. \quad (1)$$

This is the coordinate-free form of the *Cauchy-Riemann equations*.

In coordinates $z_j = x_j + iy_j$ and $f = (u_1 + iv_1, \dots, u_m + iv_m)$, equation (1) becomes:

$$\begin{cases} \partial_{x_j} u_k = \partial_{y_j} v_k \\ \partial_{y_j} u_k = -\partial_{x_j} v_k \end{cases} \quad \text{for } j = 1, \dots, n; k = 1, \dots, m.$$

A powerful reformulation uses the *Wirtinger operators*:

$$\frac{\partial}{\partial z_j} := \frac{1}{2} \left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right), \quad \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2} \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right).$$

Lemma 1.3. *The Wirtinger operators satisfy:*

$$(i) \quad \frac{\partial f}{\partial z_j} = \overline{\left(\frac{\partial \bar{f}}{\partial \bar{z}_j} \right)}$$

$$(ii) \quad \frac{\partial z_k}{\partial z_j} = \delta_{jk}, \quad \frac{\partial \bar{z}_k}{\partial z_j} = 0$$

(iii) For $f = (f_1, \dots, f_m)$ and $g = (g_1, \dots, g_n)$:

$$\begin{aligned} \frac{\partial(f \circ g)}{\partial z_j} &= \sum_{k=1}^m \left(\frac{\partial f}{\partial w_k} \frac{\partial g_k}{\partial z_j} + \frac{\partial f}{\partial \bar{w}_k} \frac{\partial \bar{g}_k}{\partial z_j} \right) \\ \frac{\partial(f \circ g)}{\partial \bar{z}_j} &= \sum_{k=1}^m \left(\frac{\partial f}{\partial w_k} \frac{\partial g_k}{\partial \bar{z}_j} + \frac{\partial f}{\partial \bar{w}_k} \overline{\left(\frac{\partial g_k}{\partial z_j} \right)} \right) \end{aligned}$$

Moreover, f is holomorphic if and only if $\frac{\partial f}{\partial \bar{z}_j} = 0$ for all j .

We can consider the complexified derivative

$$Df(z_0)^{\mathbb{C}} : T_{z_0}\mathbb{R}^{2n} \otimes \mathbb{C} \longrightarrow T_{f(z_0)}\mathbb{R}^2 \otimes \mathbb{C}.$$

The space $T_{z_0}\mathbb{R}^{2n} \otimes \mathbb{C}$ (resp. $T_{f(z_0)}\mathbb{R}^2 \otimes \mathbb{C}$) admits the canonical coordinate base $\{\partial/\partial z_i, \partial/\partial \bar{z}_i\}$ (resp. $\{\partial/\partial w, \partial/\partial \bar{w}\}$). In this base, the Jacobian in block form takes the form

The a holomorphic map f , the matrix of derivatives has the form

$$Df = \begin{pmatrix} \frac{\partial f}{\partial z_i} & 0 \\ 0 & \frac{\partial f}{\partial \bar{z}_i} \end{pmatrix},$$

reflecting complex-linearity (no $\partial/\partial \bar{z}$ -components) of f . It follows that for any holomorphic function f , $\det(Df(z_0)^{\mathbb{C}})$ is real and non-negative; $\det(Df(z_0)) \geq 0$.

Definition 1.4. A holomorphic map $f : U \rightarrow V$ is called *biholomorphic* if there exists a holomorphic inverse g to f .

If f is holomorphic and regular (non-degenerate Jacobian), then its Jacobian determinant satisfies

$$\det Df = \left| \det \left(\frac{\partial f}{\partial z_i} \right) \right|^2 > 0.$$

In particular, $\det(Df) \neq 0$ is the local invertibility criterion. Indeed, we have the holomorphic version of the inverse function theorem:

Theorem 1.5 (Holomorphic Inverse Function Theorem). *Let $U, V \subseteq \mathbb{C}^n$ open and $f : U \rightarrow V$ a holomorphic map. Consider $z_0 \in U$ such that $\det(Df(z_0)) \neq 0$. Then there exist open subsets $z_0 \in U' \subset U$ and $f(z_0) \in V' \subset V$ such that f restricts to a biholomorphism.*

More generally, a holomorphic map $f : U \rightarrow V$ is called a regular (submersion/immersion as appropriate) when the complex-linear partials $\{\partial f/\partial z_i\}_{i=1}^n$ are surjective (or injective) as needed.

Theorem 1.6 (Holomorphic Implicit Function Theorem). *Let $U \subseteq \mathbb{C}^n$ and $V \subseteq \mathbb{C}^m$ be open sets with $n > m$ and $f : U \rightarrow V$ a holomorphic function. Assume that there is z_0 such that $Df(z_0)$ satisfies*

$$\det \left[\left(\frac{\partial f_i}{\partial z_j} \right)_{i,j=1,\dots,n} \right] \neq 0. \quad (2)$$

Then there exists open sets $U_1 \subseteq \mathbb{C}^{n-m}$, $U_2 \subseteq \mathbb{C}^m$ such that $U_1 \times U_2 \subseteq U$ and a holomorphic function $g : U_1 \rightarrow U_2$ satisfying $f(w, g(w)) = f(z_0)$ for all $w \in U_1$.

Proof. The inverse function theorem guarantees the existence and differentiability of g . We need to show that g is holomorphic. Indeed, by the chain rule of Lemma 1.3, we have

$$0 = \frac{\partial}{\partial \bar{w}_j} [f_i(w, g(w))] = \frac{\partial f_i}{\partial \bar{w}_j} + \sum_{k=1}^n \frac{\partial f_i}{\partial z_k} \frac{\partial g_k}{\partial \bar{w}_j} + \frac{\partial f_i}{\partial \bar{z}_k} \overline{\left(\frac{\partial g_k}{\partial w_j} \right)} = \sum_{k=1}^n \frac{\partial f_i}{\partial z_k} \frac{\partial g_k}{\partial \bar{w}_j},$$

where the first and third terms in the middle line vanish since f is holomorphic.

But the condition in Equation (2) implies that $\left(\frac{\partial f_i}{\partial z_j} \right)$ is invertible, so the only way the second line can vanish is if $\frac{\partial g}{\partial \bar{z}_j} = 0$, as needed. $\square$

A straightforward corollary of the Holomorphic Implicit Function Theorem is the existence of left (resp. right) holomorphic inverses. We have

Corollary 1.7. *Let $U \subseteq \mathbb{C}^n$ and $V \subseteq \mathbb{C}^m$ be open sets and $f : U \rightarrow V$ a holomorphic function. Assume we have $z_0 \in U$ such that $Df(z_0)$ has maximal rank. Then,*

- (i) *If $n > m$, there exists open sets $z_0 \in U' \subset U$ and $V' \subseteq V$, and a biholomorphic map $g : V' \rightarrow U'$ such that $f \circ g = \text{Id}$ in V' .*
- (ii) *If $n < m$, there exists open sets $U' \subset U$ and $f(z_0) \in V' \subseteq V$, and a biholomorphic map $g : V' \rightarrow U'$ such that $g \circ f = (\text{Id}_n, 0)$ in U' .*

1.1 Cauchy Integral Formula and power series expansion

Recall that a key result of complex analysis is the integral formula of Cauchy:

Theorem 1.8 (Cauchy Integral Formula). *Let $K \subseteq \mathbb{C}$ be a compact subset with piecewise C^1 boundary $C = \partial K$, and $f : K \rightarrow \mathbb{C}$ a differentiable function. Then for $z \in K \setminus \partial K$, we have*

$$2\pi i f(z) = \int_{\partial K} \frac{f(w, \bar{w})}{w - z} dw + \int_K \frac{\partial f}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w - z} \quad (3)$$

Proof. Without loss of generality, we assume $z = 0$. We want to study the function $f(w, \bar{w})/w \in L^1(K)$. Taking $\delta > 0$, we have on one side,

$$\int_{K \setminus B_\delta(0)} d \left(\frac{f(w, \bar{w})}{w} \right) dz = - \int_{K \setminus B_\delta(0)} \frac{\partial f}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w}.$$

On the other side, by Stokes' theorem, we get

$$\int_{K \setminus B_\delta(0)} d \left(\frac{f(w, \bar{w})}{w} \right) dw = \int_{\partial K} \frac{f(w, \bar{w})}{w} dw - \int_{\partial B_\delta} \frac{f(w, \bar{w})}{w} dw.$$

Parametrising the last term in polar coordinates $w = \delta e^{i\theta}$, we have

$$\int_{\partial B_\delta} \frac{f(w, \bar{w})}{w} = \int_0^{2\pi} f(\delta, \theta) i d\theta.$$

Putting everything together and taking δ to zero, the claim follows by continuity of f . $\square$

Of course, we are mostly interested in the case where f is holomorphic, so the last term in (3) vanishes, and we have the usual expression

$$f(z) = \frac{1}{2\pi i} \int_{\partial K} \frac{f(w)}{w - z} dw \quad (4)$$

The Cauchy Integral Formula (CIF) generalises to higher dimensions by considering polydiscs $D_R(w) = B_{R_1}(w_1) \times \dots \times B_{R_n}(w_n)$ and iterative use of Fubini's theorem.

Exercise 1.9. *Prove the n -dimensional Cauchy Integral Formula in detail:*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial D_R(z)} \frac{f(w_1, \dots, w_n)}{(w_1 - z_1) \dots (w_n - z_n)} dw_1 \dots dw_n.$$

The CIF has some important, remarkable consequences for the regularity of the function f :

Proposition 1.10. *Let $f : U \rightarrow \mathbb{C}$ be a holomorphic function. Then f is analytic. That is, it admits a convergent power series expansion*

$$2\pi i f(z) = \sum_{|\alpha| \geq 0} \frac{f^{(\alpha)}(z_0)}{\alpha!} z^\alpha ,$$

with α a multi-index $\alpha = (\alpha_1, \dots, \alpha_n), .$

Proof. We argue the case $n = 1$; the higher-dimensional case follows. We know $\frac{1}{z-w} = \frac{1}{z} \frac{1}{(1-w/z)} = \sum_{k \geq 0} \frac{w^k}{z^{k+1}}$ for $|w| < |z|$. Substituting in the CIF and using Lebesgue monotone convergence, we have

$$2\pi i f(w) = \int_C \frac{f(z)}{z-w} dz = \int_C \sum_{k \geq 0} w^k \frac{f(z)}{z^{k+1}} dz = \sum_{k \geq 0} w^k \int_C \frac{f(z)}{z^{k+1}} dz .$$

Analyticity follows. The coefficients of the power expansion are the successive derivatives of f by the uniqueness of Taylor expansions. Alternatively, one can check directly:

$$\begin{aligned} f'(w) &= \lim_{h \rightarrow 0} \frac{f(w+h) - f(w)}{h} = \lim_{h \rightarrow 0} \frac{1}{2\pi i h} \int_C \frac{f(z)}{z - (w+h)} - \frac{f(z)}{z-w} dz \\ &= \lim_{h \rightarrow 0} \frac{1}{2\pi i h} \int_C \frac{h f(z)}{(z-w-h)(z-w)} dz = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-w)^2} dz . \end{aligned} \quad \square$$

The analyticity of holomorphic functions has some remarkable consequences:

Theorem 1.11 (Open mapping theorem). *Let $f : U \rightarrow \mathbb{C}$ be a non-constant holomorphic function on an open set U . The f is an open mapping.*

In particular, if there exists $z_0 \in U$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in U$, f is constant.

Theorem 1.12 (Identity principle). *Let U be an open connected subset of $\mathbb{C}^n$ and $f, g : U \rightarrow \mathbb{C}$ holomorphic functions. If $f = g$ on an open subset $V \subset U$, then $f \equiv g$ on all of U .*

Proof. Let

$$W = \left\{ z \in U \mid \frac{\partial^\alpha f}{\partial z^\alpha} = \frac{\partial^\alpha g}{\partial z^\alpha} \quad \forall \alpha \text{ multi-index} \right\} .$$

The set W is clearly closed and non-empty. By analyticity, W is also open, and by connectedness, $W = U$. $\square$

Another consequence of the Cauchy Integral Formula, Equation (4), is

Lemma 1.13 (Cauchy inequality). *Let $f : U \rightarrow \mathbb{C}$ be a holomorphic function and take $R > 0$ such that the ball $B_R(z_0)$ is contained in U . Then*

$$|f^{(\alpha)}(z_0)| \leq \frac{\alpha!}{R^\alpha} \sup_{\partial B_R(z_0)} |f(z)| \quad (5)$$

There are two important corollaries of this inequality:

Theorem 1.14 (Generalised Liouville theorem). *Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ a holomorphic function such that $|f(z)| \leq C(1 + |z|)^D$ for some $C, D \geq 0$. Then f is a polynomial with degree at most D .*

Theorem 1.15 (Montel's theorem). *Let $U \subseteq \mathbb{C}^n$ open, and consider $\mathcal{O}(U)$ the space of holomorphic functions on U , equipped with the uniform convergence on compact sets topology, induced by $C^0(U)$. Then every locally uniformly bounded sequence $(f_j)_j \subseteq \mathcal{O}(U)$ has a convergent subsequence.*

Proof. By Arzelà–Ascoli. □

1.2 Hartogs' phenomenon and the Weierstrass theorems

So far, all properties that we have discussed are direct analogues of properties that occur in complex analysis ($n = 1$) and have discussed the rigidity of holomorphic functions. First, we need the following technical lemma

Lemma 1.16. *Consider the open cylinder $U \times V$ with $U \subseteq \mathbb{C}^n$ open, and $V \subseteq \mathbb{C}$ a neighbourhood of $\partial B_\varepsilon(z_0)$ and let $f : V \times U \rightarrow \mathbb{C}$ a holomorphic function. Then*

$$g(z_1, \dots, z_n) := \int_{\partial B_\varepsilon(z_0)} f(\xi, z_1, \dots, z_n) d\xi$$

is a holomorphic function on U .

Proof. Notice that if f were holomorphic on $U \times B_\varepsilon(z_0)$, we would essentially be done. The idea is to reduce it to an equivalent situation.

Since $\partial B_\varepsilon(z_0)$ is compact, for every $\delta > 0$, there exists finitely many ξ_i such that $\{B_\delta(\xi_i)\}$ cover $\partial B_\varepsilon(z_0)$. By choosing δ small enough, we can ensure $B_\delta(\xi_i) \subseteq V$ and f has a convergent power series in $B_\delta(\xi_i) \times U_i$ for all i .

We can now split the integral into a finite sum of integrals where f has a power series expansion. □

Let us now focus on the extension problem.

Theorem 1.17 (Hartogs' principle). *Let $D_R(0)$ and $D_{R'}(0)$ be two polydiscs in $\mathbb{C}^n$ with $\overline{D_{R'}(0)} \subseteq D_R(0)$ so $R_i > R'_i$ for all i . Any holomorphic function $f : D_R(0) \setminus \overline{D_{R'}(0)} \rightarrow \mathbb{C}$ can be uniquely extended to a holomorphic function $\bar{f} : D_R(0) \rightarrow \mathbb{C}$.*

Proof. Let $w = (z_2, \dots, z_n)$ with $|z_2| > R'_2$. We can use the Cauchy formula for the function $z \mapsto f(z, w)$, for $R'_1 < \delta < R_1$:

$$f(z, w) = \frac{1}{2\pi i} \int_{|\xi|=\delta} \frac{f(\xi, w)}{(\xi - z)} d\xi$$

The integrand is $(\xi, z, w) \mapsto \frac{f(\xi, w)}{(\xi - z)} d\xi$, which is holomorphic on $B_c(\delta) \times B_{\delta-c}(0) \times D_{R_2, \dots, R_n}(0)$ for some small c . Therefore, by the lemma, the function

$$\tilde{f}(z, w) = \frac{1}{2\pi i} \int_{|\xi|=\delta} \frac{f(\xi, w)}{(\xi - z)} d\xi$$

is holomorphic on $B_{\delta-c}(0) \times D_{R_2, \dots, R_n}$, providing the desired extension by the identity principle. □

We conclude this subsection by proving two technical lemmas, due to Weierstrass, that will be useful throughout the course. First, we need

Definition 1.18 (Weierstrass Polynomial). A *Weierstrass polynomial* in z_1 of degree d is a polynomial

$$z_1^d + a_1(w)z_1^{d-1} + \cdots + a_d(w),$$

where $a_i(z')$ are holomorphic functions in $w = (z_2, \dots, z_n)$ defined in a neighbourhood of the origin and such that $a_i(0, \dots, 0) = 0$.

Theorem 1.19 (Weierstrass Preparation Theorem). *Let $f : D_\varepsilon(0) \rightarrow \mathbb{C}$ with $f(0,0) = 0$ and $f(z_1, 0, \dots, 0) \not\equiv 0$. Then for some smaller ball $D_{\varepsilon'}(0)$ there exists a unique decomposition:*

$$f = g \cdot h$$

where g is a Weierstrass polynomial in z_1 , and $h : D_{\varepsilon'}(0) \rightarrow \mathbb{C}$ is a holomorphic function without zeros.

Proof. By taking ε_1 smaller if needed, we may assume $f(z_1, 0, \dots, 0)$ vanishes only at 0, with multiplicity d . Moreover, choose $r \in (0, \varepsilon)$ and $\varepsilon_2, \dots, \varepsilon_n$ so that $f(z_1, w) \neq 0$ for $|z_1 - r| < \varepsilon$ and $|w_i| < \varepsilon_i$, which exist by continuity and compactness.

For small w , the zeros of $f_w(z) = f(z, w)$ are given by $a_1(w), \dots, a_d(w)$. Define:

$$g(z, w) = \prod_{i=1}^d (z_1 - a_i(w)), \quad h = \frac{f}{g}$$

We need to show that g and h are holomorphic in z_1 and w . Holomorphicity in z_1 is straightforward. To see g is holomorphic in w , notice that this amounts to showing that the elementary symmetric polynomials in terms of $a_i(w)$ are holomorphic, which are linear combinations of $S_k = \sum_{i=1}^n a_i(w)^k$ for $k = 0, \dots, d$. By the Cauchy residue formula¹, we have

$$\sum_{i=1}^n a_i(w)^k = \frac{1}{2\pi i} \int_{|\xi|=\varepsilon_1} \xi^k \frac{\partial}{\partial \xi} \left[\log(f(\xi, w)) \right] d\xi,$$

which is holomorphic by Lemma 1.16. Finally, we may write

$$h(z_1, w) = \frac{1}{2\pi i} \int_{|\xi|=\varepsilon'_1} \frac{h(\xi, w)}{\xi - z_1} d\xi,$$

which is everywhere holomorphic by Lemma 1.16 and f/g being holomorphic on the annulus. $\square$

Theorem 1.20 (Weierstrass Division Theorem). *Let $f \in \mathcal{O}_{\mathbb{C}^n, 0}$, and let g be a Weierstrass polynomial of degree d . Then there exist a unique $h \in \mathcal{O}_{\mathbb{C}^n, 0}$ and $r \in \mathcal{O}_{\mathbb{C}^{n-1}, 0}[z_1]$ with $\deg r < d$ such that:*

$$f = g \cdot h + r$$

Proof. Define

$$h(z, w) = \frac{1}{2\pi i} \int_{\partial B_\varepsilon(0)} \frac{f(\xi, w)}{g(\xi, w)} \frac{d\xi}{\xi - z}$$

and check that $r = f - gh$ lies in $\mathcal{O}_{\mathbb{C}^{n-1}, 0}[z_1]$ and is of degree $< d$ holomorphicity. $\square$

¹Check this formula by yourself, note that $k = 0$ is precisely the argument principle, giving the count of zeros enclosed in the domain.

1.3 The ring of holomorphic germs $\mathcal{O}_{\mathbb{C}^n,0}$ and Hilbert's Nullstellensatz

We study the local behaviour of holomorphic functions on an arbitrarily small neighbourhood of a point. More formally, this leads to considering the notion of germs and stalks:

Definition 1.21. The *holomorphic stalk at the origin*, denoted $\mathcal{O}_{\mathbb{C}^n,0}$, is the set of all equivalence classes of pairs (U, f) , where U is an open neighbourhood of 0 in $\mathbb{C}^n$ and $f : U \rightarrow \mathbb{C}$ is a holomorphic function.

Two pairs (U, f) and (V, g) are considered equivalent if there exists an open neighbourhood $W \subseteq U \cap V$ of 0 such that f and g agree on W :

$$(U, f) \sim (V, g) \iff f|_W = g|_W \text{ for some open } W \ni 0.$$

An equivalence class is called a *holomorphic germ at 0*.

Alternatively, one can think of the holomorphic stalk as the set of convergent power series inside $\mathbb{C}[[z_1, \dots, z_n]]$.

Exercise 1.22. Prove that this is indeed the case, i.e. there is a one-to-one correspondence between convergent power series and holomorphic germs.

Remark 1.23. Definition 1.21 might feel overly complicated and slightly unnatural. Indeed, stalks and germs are better understood in the language of sheaves, which we will introduce in Section 3.

The holomorphic stalk $\mathcal{O}_{\mathbb{C}^n,0}$ inherits a ring structure from that of holomorphic functions. We devote ourselves to studying its structure. We shall prove

Theorem 1.24. *The stalk of holomorphic germs $\mathcal{O}_{\mathbb{C}^n,0}$ is*

- (i) *a local ring,*
- (ii) *a unique factorisation domain (UFD), and*
- (iii) *Noetherian.*

Proof. (i) The ideal $\mathcal{I}_0$ given by (germs of) functions vanishing at the origin is maximal, with residue field $\mathcal{O}_{\mathbb{C}^n,0}/\mathcal{I}_0 \cong \mathbb{C}$. If $f \in \mathcal{O}_{\mathbb{C}^n,0}$ satisfies $f \neq 0$, then one can show with little work that $f \in \mathcal{O}_{\mathbb{C}^n,0}^*$, so there's no other maximal ideal $\mathcal{I}_0$.

(ii) We prove this by induction. The case $n = 0$ is trivial.

Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$ vanishing at the origin. By the Weierstrass Preparation Theorem 1.19, we can uniquely write f as $f = u \cdot p$, with $u \in \mathcal{O}_{\mathbb{C}^n,0}^\times$ a unit and $p \in \mathcal{O}_{\mathbb{C}^{n-1},0}[w]$ (the germ of) a Weierstrass polynomial.

The $\mathcal{O}_{\mathbb{C}^{n-1},0}$ is a UFD by induction hypothesis, and so is $\mathcal{O}_{\mathbb{C}^{n-1},0}[w]$ by Gauss' lemma.

It remains to check that p is a finite irreducible element of $\mathcal{O}_{\mathbb{C}^n,0}$, which is straightforward using the uniqueness of the decomposition of the Weierstrass Preparation Theorem 1.19.

(iii) Again, we prove this by induction, with the case $n = 0$ being immediate.

Assume $\mathcal{O}_{\mathbb{C}^{n-1},0}$ is Noetherian, and therefore so is the subring $\mathcal{O}_{\mathbb{C}^{n-1},0}[z_1] \subseteq \mathcal{O}_{\mathbb{C}^n,0}$, by Hilbert's basis theorem.

Let $I \in \mathcal{O}_{\mathbb{C}^n,0}$ an ideal, so $I \cap \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ is finitely generated.

Take $f \in I$. By the Weierstrass Preparation Theorem 1.19, we get $f = gh$ with $h \in \mathcal{O}_{\mathbb{C}^n,0}^*$ and $g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$, so $g = fh^{-1} \in I \cap \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$.

For any other $\tilde{f} \in I$, the Weierstrass division theorem implies that $\tilde{f} = g\tilde{h} + \tilde{r}$ for $r \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$. Since $\tilde{f}$ and g are in I , it follows that $r \in I \cap \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$. Thus, I is a finitely generated ideal.

□

We include one final lemma for the sheaf of holomorphic stalks that will be useful in the future:

Lemma 1.25. *Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$ irreducible. Then for $\varepsilon > 0$ small enough $f \in \mathcal{O}_{\mathbb{C}^n,z}$ is irreducible for all $z \in B_\varepsilon(0)$. Similarly, if $f, g \in \mathcal{O}_{\mathbb{C}^n,0}$ are coprime, they remain coprime in $\mathcal{O}_{\mathbb{C}^n,z}$ for all $z \in B_\varepsilon(0)$ for ε small enough.*

Proof. We include the details for the proof of when f and g are coprime; the proof of irreducibility follows the same logic.

By the Weierstrass Preparation Theorem 1.19, we may assume f and g are Weierstrass polynomials. Thus, they must be coprime as polynomials. By Gauss' lemma, this means we can find polynomials $p_1, p_2 \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ and $0 \neq h \in \mathcal{O}_{\mathbb{C}^{n-1},0}$ such that $h = fp_1 + gp_2$. The claim follows. □

Let us now define analytic sets and their germs. Given $f : U \rightarrow \mathbb{C}$ a holomorphic function, we denote its vanishing set as $Z(f) = \{z \in U \mid f(z) = 0\}$.

Definition 1.26. An *analytic set* $Z \subseteq X$ is a set such that for each $x \in Z$, there exists an open neighbourhood $U \ni x$ and holomorphic functions $f_1, \dots, f_k \in \mathcal{O}(U)$ with

$$Z \cap U = Z(f_1, \dots, f_k) = \bigcap_{i=1}^k Z(f_i).$$

In the same spirit as before, we define the corresponding germs

Definition 1.27. An *analytic germ* at $x \in X$ is an equivalence class of analytic sets under the relation $Z_1 \sim Z_2$ if $Z_1 \cap U = Z_2 \cap U$ for some neighbourhood $U \ni x$.

Given a germ X at the origin, we denote by $I(X)$ the set of holomorphic germs s satisfying the condition $X \subseteq Z(s)$. So $Z(\cdot)$ takes holomorphic germs (or functions) to analytic germs, and $I(\cdot)$ takes analytic germs to their holomorphic counterparts. They satisfy the following relations:

Lemma 1.28.

- (i) For any subset $A \subseteq \mathcal{O}_{X,x}$, $Z(A)$ is a well-defined analytic germ with $Z(A) = Z((A)_{\mathcal{O}_{X,x}})$.
- (ii) For every analytic germ Z , $I(Z) = \{f \in \mathcal{O}_{X,x} \mid Z \subset Z(f)\}$ is an ideal.

(iii) If $X_1 \subset X_2$ are analytic germ, then $I(X_2) \subset I(X_1)$. If $I_1 \subset I_2$ are ideals in $\mathcal{O}_{X,x}$, then $Z(I_2) \subset Z(I_1)$.

(iv) $Z = Z(I(Z))$ and $I \subset I(Z(I))$.

(v) $Z(I \cdot J) = Z(I) \cup Z(J)$ and $Z(I + J) = Z(I) \cap Z(J)$.

Proof. Exercise. □

The relation between holomorphic and analytic germs is made precise by Hilbert's Nullstellensatz:

Theorem 1.29 (Hilbert's Nullstellensatz Theorem). *For any ideal $I \subseteq \mathcal{O}_{X,x}$, we have:*

$$\sqrt{I} = I(Z(I))$$

where $\sqrt{I}$ is the radical ideal of I ; $\sqrt{I} = \{f \in \mathcal{O}_{X,x} \mid f^n \in I \text{ for some } n\}$.

We would like to understand the fundamental “building blocks” of holomorphic and analytic germs. Since the holomorphic stalk naturally carries a ring structure, our focus will be on its prime ideals. On the side of analytic germs, we introduce the following definition:

Definition 1.30. An analytic germ is Z called *irreducible* if for any union $Z = Z_1 \cup Z_2$ with Z_i analytic germs, either $Z = Z_1$ or $Z = Z_2$.

As expected, we have the following result

Lemma 1.31. *An analytic germ Z is irreducible if and only if $I(Z)$ is a prime ideal.*

Proof. Let $f_1 f_2 \in I$. Then $Z = (Z \cap Z(f_1)) \cup (Z \cap Z(f_2))$. If Z is irreducible $Z = Z \cap Z(f_i)$, so f_i vanishes along Z , i.e. $f_i \in I(Z)$.

The converse follows similarly. □

2 Complex and almost complex manifolds

We now introduce the main class of objects that we are interested in, complex manifolds. We will give two definitions for them. First, using complex charts and holomorphic transition functions. Second, we adopt a more differential geometric style, using $\text{GL}(n, \mathbb{C})$ -structures, more commonly known as almost complex structures on a real manifold. The two definitions are equivalent by virtue of the celebrated Newlander-Nirenberg Theorem.

For the remainder of the notes, a (topological) manifold is a locally Euclidean, second-countable ², Hausdorff space. Recall from differential geometry:

Definition 2.1. A $\mathcal{C}^k$ -manifold is a topological manifold equipped with an atlas of charts $(U_i, \phi_i)_{i \in I}$, where transition functions $\phi_{ij} = \phi_i \circ \phi_j^{-1}$ are $\mathcal{C}^k$ -diffeomorphisms between open sets in $\mathbb{R}^n$.

²Sometimes

Recall that $\mathcal{C}^0$ -manifolds are topological manifolds, and that a theorem of Whitney tells us that a $\mathcal{C}^k$ -manifold for $k \geq 1$ admits a compatible $\mathcal{C}^\infty$ -structure.

There is an intermediate notion between $\mathcal{C}^0$ and $\mathcal{C}^1$, called PL :

Understanding when a manifold admits a smooth structure, and if so, how many, was an active research area in the second half of the 20th century that is nowadays well understood (see e.g. Kervaire–Milnor groups, Kirby–Siebenmann invariants, geometrisation conjecture) except in dimension 4, where surprising links to other areas of mathematics appear.

Another class (before I digress too much) is the class of affine manifolds, where the $\mathcal{C}^k$ condition is replaced by $\text{Aff}(\mathbb{R}^n)$, requiring the transition maps to be affine maps of $\mathbb{R}^n$. Affine manifolds are quite mysterious, and longstanding conjectures and open problems remain to be tackled.

Definition 2.2. A *complex manifold* is a manifold equipped with an atlas of charts $(U_i, \phi_i)_{i \in I}$, where transition functions $\phi_{ij} = \phi_i \circ \phi_j^{-1}$ are biholomorphisms between open sets in $\mathbb{C}^n$.

To avoid issues and pathologies, we will always assume our atlases are maximal, i.e. they are not a proper subset of any other atlas. Every atlas $\{(U_i, \phi_i) : i \in I\}$ is contained in a unique maximal atlas: the set of all charts (U, ϕ) compatible with (U_i, ϕ_i) for all $i \in I$, so there is no prejudice in always taking the maximal atlas.

We will mostly refer to X as the complex manifold, omitting the atlas to lighten notation, as is typically done in differential geometry. As in the previous case, we can ask the questions:

Question 2.3. *When does a manifold M admit the structure of a complex manifold? Is the complex structure unique? Can we classify complex manifolds up to biholomorphism?*

In contrast to the smooth case, very little is known in this case, beyond some obvious topological constraints, discussed in the exercises.

In the compact setting, some existence and classification results exist for complex dimensions 1 and 2. Already in dimension 3, we find one of the most (in)famous open problems in differential geometry:

Question 2.4. *Does the round 6-sphere S^6 admit the structure of a complex structure?*

In the non-compact case, we have Liouville-type obstructions, so we know that the complex plane $\mathbb{C}^n$ is not biholomorphic to certain bounded domains (e.g. the unit ball or polydisc). However, there is no high-dimensional analogue of the Uniformisation Theorem. In general, complex domains carry intrinsic complex-analytic invariants that obstruct biholomorphism. For $n > 1$, many bounded domains are not biholomorphically equivalent.

Definition 2.5. Let X be a complex manifold, and $f : X \rightarrow \mathbb{C}$ a function. We call f *holomorphic* if, for all charts (U, ϕ) in the (maximal) atlas, $f \circ \phi$ is holomorphic in the sense of Section 1.

Definition 2.6. Let X, Y be complex manifolds and $f : X \rightarrow Y$ a continuous function. The map f is said to be *holomorphic* if for all charts (U, ϕ) of X and (V, ψ) of Y , the map

$$\psi^{-1} \circ f \circ \phi$$

is a holomorphic map in the sense of Section 1.

Definition 2.7. Let X be a complex manifold of dimension n , and $Y \subseteq X$.

We say Y is an (embedded) *complex submanifold* of X of dimension k if for each $y \in Y$ there exist an open neighbourhood U of y and local holomorphic coordinates $(z_1, \dots, z_n)$ on U such that $Y = Z(z_{k+1}, \dots, z_n)$.

We will usually require Y to be closed in X . With the definition above, it is easy to see that

Proposition 2.8. *A complex submanifold is a complex manifold such that the inclusion map $\iota_Y : Y \hookrightarrow X$ is injective and holomorphic.*

Conversely, a holomorphic map $f : Y \rightarrow X$ is called an embedding if it is injective, locally closed, and with injective differential $Df : T_y Y \rightarrow T_{f(y)} X$ for all $y \in Y$. It follows easily that f is an embedding if and only if $f(Y)$ is a complex submanifold of X , biholomorphic to Y .

As in the smooth case, we can produce examples of complex submanifolds via the holomorphic implicit function theorem:

Theorem 2.9. *Let $f : X \rightarrow Y$ be a holomorphic map between complex manifolds of dimensions n and m respectively, and let $y \in Y$ such that the differential $Df_x : T_x X \rightarrow T_y Y$ is surjective for all $x \in f^{-1}(y)$. Then $f^{-1}(y)$ is a complex submanifold of dimension $n - m$.*

A point y satisfying the conditions of the theorem above is called a *regular point* (or value, if $Y = \mathbb{C}$). We have

Corollary 2.10. *Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be a holomorphic function and c a regular value, then $Z(f - c) = f^{-1}(c)$ is a complex hypersurface (complex submanifold) of complex codimension 1.*

Unfortunately, one needs to work a bit harder if one is interested in finding examples of compact complex submanifolds.

Exercise 2.11. *The only compact complex submanifolds of $\mathbb{C}^n$ (when considered as submanifolds of $\mathbb{C}^n$) are discrete points.*

Let us introduce the first compact example, which will play a prominent role throughout the course. The complex projective space $\mathbb{CP}^n$ is the moduli space of complex lines (or dually hyperplanes) in $\mathbb{C}^{n+1}$. It can be realised as the quotient

$$\mathbb{CP}^n \cong (\mathbb{C}^{n+1} \setminus \{0\}) / \mathbb{C}^*,$$

where the $\mathbb{C}^*$ -action is given by $z \mapsto \lambda z$.

The complex projective space $\mathbb{CP}^n$ is a compact n -dimensional complex manifold.

Let us define homogeneous coordinates $[z_0, \dots, z_n]$ on $\mathbb{CP}^n$. For $i = 0, \dots, n$, define a chart (U_i, ϕ_i) on $\mathbb{CP}^n$ by $U_i = \mathbb{C}^n$ and $\phi_i : \mathbb{C}^n \rightarrow \mathbb{CP}^n$ given by

$$\phi_i : (w_1, \dots, w_n) \mapsto [w_1, \dots, w_i, 1, w_{i+1}, \dots, w_n].$$

This is a homeomorphism with the open subset

$$\phi_i(U_i) = \{[z_0, \dots, z_n] \in \mathbb{CP}^n : z_i \neq 0\} \text{ in } \mathbb{CP}^n.$$

For $0 \leq i < j \leq n$, the transition function $\phi_{ij} = \phi_j^{-1} \circ \phi_i$ is given by

$$\begin{aligned} \phi_{ij} : \mathbb{C}^n \setminus \{z_j = 0\} &\rightarrow \mathbb{C}^n \setminus \{z_i = 0\} \\ (z_1, \dots, z_n) &\mapsto \left(\frac{z_1}{z_j}, \dots, \frac{z_i}{z_j}, \frac{1}{z_j}, \frac{z_{i+1}}{z_j}, \dots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \dots, \frac{z_n}{z_j} \right). \end{aligned}$$

The ϕ_{ij} 's are clearly biholomorphisms. So $\{(U_i, \phi_i)\}_{i=0, \dots, n+1}$ forms an atlas of $\mathbb{CP}^n$, that extends to the corresponding maximal atlas.

Now, we have the following example of complex submanifolds:

Proposition 2.12. *Let $p : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{C}$ a homogeneous polynomial such that 0 is a regular value of p , and consider*

$$X = \{[z_0, \dots, z_n] \in \mathbb{CP}^n \mid (z_0, \dots, z_n) \in Z(p)\}.$$

Then X is a well-defined compact complex submanifold of $\mathbb{CP}^n$.

Proof. X is well-defined, since p is homogeneous, so $p(z) = 0$ implies $p(\lambda z) = 0$ for all $\lambda \in \mathbb{C}^*$.

Now, X is covered by the charts $V_i = (X \cap U_i)$, where U_i are the standard charts for $\mathbb{CP}^n$ used above. On each V_i , X is described by the vanishing of $p(z_0, \dots, z_{i-1}, 1, z_{i+1}, \dots, z_n)$, and Theorem 2.9 concludes the proof. $\square$

We give two examples:

Example 2.13. *For $d \in \mathbb{N}_+$, the set $X = (z_0^d + z_1^d + z_2^d) \subseteq \mathbb{CP}^2$ is a Riemann surface of genus $g = \frac{(d-1)(d-2)}{2}$.*

Example 2.14. *The set $Y = Z(z_0^2 + \dots + z_3^2) \subseteq \mathbb{CP}^3$ is a projective complex manifold biholomorphic, $\mathbb{CP}^1 \times \mathbb{CP}^1$.*

Of course, one may ask how general the condition for 0 to be a regular value of a homogeneous polynomial. We leave it as an exercise to show that

Exercise 2.15. *The set of homogeneous polynomials for which 0 is a regular value is generic.*

More generally, one has

Proposition 2.16. *Let $(p_1, \dots, p_k) : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{C}^k$ a collection of homogeneous polynomials such that $(0, \dots, 0)$ is a regular value. Then $(Z(p_1) \cap \dots \cap Z(p_k))/\mathbb{C}^* \subseteq \mathbb{CP}^n$ is a complex submanifold of dimension $n - k$, called a complete intersection.*

More generally, a projective variety is a subset X of $\mathbb{CP}^n$ which is locally defined by the vanishing of finitely many homogeneous polynomials.

Projective complex manifolds allow us to consider a large number of examples of complex manifolds. Moreover, since they are defined using polynomials, they can be studied using algebraic techniques, giving rise to complex algebraic geometry.

In the opposite direction, one may consider under what conditions one can guarantee that a compact complex manifold X can be realised as a projective complex manifold. The answer to this question is fully understood and follows from two important results, Chow's Theorem and the Kodaira Embedding Theorem, which we will prove during this course.

Complex Lie groups also provide important examples of complex manifolds:

Definition 2.17. A *complex Lie group* is a group G that is also a complex manifold such that multiplication and inversion are holomorphic maps.

Examples include the general linear groups $\mathrm{GL}_n(\mathbb{C})$, special linear groups $\mathrm{SL}_n(\mathbb{C})$, complex tori, etc.

Proposition 2.18. *Let G be a complex Lie group acting holomorphically on a complex manifold X . If the action is free and proper, then the quotient X/G carries a canonical complex manifold structure for which the projection $X \rightarrow X/G$ is a holomorphic submersion.*

Proof. See [Wel08, Prop. 5.3]. □

As a direct application of this proposition, we give two further examples of complex manifolds: Hopf and Iwasawa manifolds.

Hopf manifolds are examples of compact complex manifolds obtained as quotients of $\mathbb{C}^n \setminus \{0\}$ by a discrete group generated by contractions. For a concrete example, let $\alpha \in (0, 1)$ and

$$H_A = (\mathbb{C}^n \setminus \{0\}) / \sim_\alpha$$

where $z \sim_\alpha w$ if $z = \alpha^n w$ for some n .

Remark 2.19. Hopf manifolds are diffeomorphic to $S^{2n-1} \times S^1$ (think in polar coordinates) and provide important examples in complex geometry, as we shall see.

Finally consider $\mathbb{U} \subseteq \mathrm{GL}(3, \mathbb{C})$ the subgroup of upper-triangular matrices

$$U = \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix}$$

and its subgroup $\mathbb{U}_{\mathbb{Z}} = \mathbb{U} \cap \mathrm{GL}(3, \mathbb{Z}[i])$. The group $\mathbb{U}_{\mathbb{Z}}$ acts by translations $(w_1, w_2, w_3) \cdot (z_1, z_2, z_3) \mapsto (z_1 + w_1, z_2 + w_2, z_3 + w_3)$, which is a free and proper action, so the quotient is a complex manifold, known as the Iwasawa manifold $\mathbb{I} = \mathbb{U} / \mathbb{U}_{\mathbb{Z}}$.

The first and third coordinate provide a holomorphic submersion $f : \mathbb{I} \rightarrow \mathbb{C}/\mathbb{Z}[i] \times \mathbb{C}/\mathbb{Z}[i]$, with the fibres given by the remaining coordinate, biholomorphic to $\mathbb{C}/\mathbb{Z}[i]$.

2.1 Almost complex structures

We now introduce the second definition of complex manifolds, via almost complex structures. The idea is to consider a weaker notion of complex structures and study the relation between the two. The idea is the following: Let X be a complex n -manifold in the sense of Definition 2.2. Then, the underlying topological manifold carries a natural smooth real $2n$ -manifold $X_{\mathbb{R}}$. Its tangent bundle $TX_{\mathbb{R}}$ inherits the structure of a complex vector bundle, which is reflected in the existence of a bundle endomorphism $J \in \mathcal{C}^\infty(\mathrm{End}(TX_{\mathbb{R}}))$ such that $J^2 = -\mathrm{Id}_{2n}$ fiberwise. This motivates the notion of an almost complex structure:

Definition 2.20. Let X be a real $2n$ -manifold. An *almost complex structure* J on X is the choice of a section J in $\mathcal{C}^\infty(\mathrm{End}(TX_{\mathbb{R}}))$ satisfying the condition $J^2 = -\mathrm{Id}_{2n}$.

A manifold X equipped with an almost complex structure J is called an *almost complex manifold*.

Any complex manifold in the sense of Definition 2.2 induces a real manifold X with an almost complex structure J . The converse is not true, as we shall see.

Since an almost complex structure J furnishes the tangent space with the structure of a complex vector space pointwise, we can define the analogue notions of holomorphic functions and maps.

Definition 2.21. Let (X, J) be an almost complex manifold and $f : X \rightarrow \mathbb{C}$ a smooth function. We say f is *J-holomorphic* function if

$$df \circ J = idf .$$

Similarly, we have

Definition 2.22. Let (X, I) and (Y, J) be almost complex manifolds and $f : X \rightarrow Y$ a smooth map. We say f is a *pseudo-holomorphic* map if

$$df \circ I = J \circ df .$$

Before proceeding, let us say a few words about the existence of almost complex structures.

Unlike the case of complex structures, we are not requiring that our structure solves any PDEs (the transition maps being holomorphic), just the existence of a special section of the endomorphism bundle $\text{End}(TM)$ (or the reduction of the frame bundle to a principal $\text{GL}(n, \mathbb{C})$ -bundle). This problem is well-understood from the point of view of classifying spaces, and it allows us to phrase necessary and sufficient conditions for the existence of an almost complex structure in terms of very explicit topological conditions in low dimensions:

Proposition 2.23. *Let M^{2n} be a closed manifold*

- (i) *For $n = 1$, M admits an almost complex structure if and only if M is orientable (equiv. $w_1(M) = 0$).*
- (ii) *For $n = 2$, M admits an almost complex structure if and only if M is orientable and there exists $h \in H^2(M, \mathbb{Z})$ such that*

$$h^2 = 3\sigma(X) + 2\chi(X) \qquad h \equiv_2 w_2(X) .$$

We refer the interested reader to [MS74, §12] for an introductory discussion on obstruction theory on vector bundles.

2.2 The exterior differential and the Nijenhuis tensor

Let us now explore the geometry of almost complex manifolds. For the remainder of the section (X^n, J) will denote an almost complex manifold of (complex) dimension n .

Lemma 2.24. *The complexified tangent bundle $TX_{\mathbb{C}} := TX \otimes_{\mathbb{R}} \mathbb{C}$ splits as a direct sum of complex bundles $TX^{1,0} \oplus TX^{0,1}$ of complex dimension n , given by*

$$TX^{1,0} = \ker (i \text{Id} - J) \qquad TX^{0,1} = \ker (i \text{Id} + J)$$

Proof. The minimal polynomial of J is $x^2 - 1 = (x - i)(x + i)$, which means J is diagonalisable over $\mathbb{C}$. The bundles $TX^{1,0}$ and $TX^{0,1}$ are the corresponding eigenbundles $\square$

Remark 2.25. While $TX^{1,0}$ and $TX^{0,1}$ are not in general isomorphic as complex bundles, they are always isomorphic as real bundles, with the isomorphism given by conjugation.

The decomposition of the complexified tangent bundle into holomorphic and anti-holomorphic parts trickles down into all associated vector bundles. In particular, we have the following decomposition of exterior k -forms:

$$\bigwedge^k T^*M \otimes \mathbb{C} = \bigoplus_{p+q=k} \bigwedge^{p,q} T^*M \quad \bigwedge^{p,q} T^*M := \bigwedge^p (T^*X^{1,0}) \otimes \bigwedge^q (T^*X^{1,0}) .$$

We denote the space of smooth sections of $\bigwedge^{p,q} T^*M$ by $\mathcal{A}^{p,q} = \Gamma(X, \bigwedge^{p,q} T^*M)$.

There is a more abstract way of understanding this decomposition. An almost complex carry a reductio of the structure group $\mathrm{GL}(n, \mathbb{C}) \subset \mathrm{GL}(2n, \mathbb{R})$, and the decomposition of k -forms into (p, q) -forms corresponds to decomposition of $\Lambda^k(\mathbb{R}^{2n})^* \otimes_{\mathbb{R}} \mathbb{C}$ into irreducible representations of $\mathrm{GL}(n, \mathbb{C})$.

We can study how the exterior differential behaves with respect to this decomposition. We have the following:

Proposition 2.26. *There exists operators $\partial : \mathcal{A}^{p,q} \rightarrow \mathcal{A}^{p+1,q}$ and $\mu : \mathcal{A}^{p,q} \rightarrow \mathcal{A}^{p+2,q-1}$ such that the exterior differential d decomposes as*

$$d = \mu + \partial + \bar{\partial} + \bar{\mu} ,$$

with $\bar{\partial}$ and $\bar{\mu}$ are the conjugate operators to ∂ and μ respectively.

Proof. The exterior differential d is a local operator. Any (p, q) -form γ can be written down locally as

$$\gamma = \sum_{|I|=p, |J|=q} f_{I,J} \alpha^I \wedge \bar{\alpha}^J$$

with $\{\alpha_1, \dots, \alpha_n\}$ a local basis of $\mathcal{A}^{1,0}$. $\square$

Lemma 2.27. *The operators ∂ and μ satisfy the following properties:*

- (i) the Leibniz rule,
- (ii) ∂ is $\mathbb{C}$ -linear and μ is function linear, and
- (iii) the following identities hold:

$$\begin{aligned} \mu\partial + \partial\mu &= 0 , & \partial^2 + \bar{\partial}\mu + \mu\bar{\partial} &= 0 , \\ \mu^2 &= 0 , & \mu\bar{\mu} + \bar{\partial}\partial + \partial\bar{\partial} + \bar{\mu}\mu &= 0 . \end{aligned}$$

Proof. Exercise. $\square$

Since μ is function-linear, we can identify the operator μ acting on $(0,1)$ -forms with a tensor $N_J \in \Gamma(X, \text{Hom}(T^*X^{0,1}, \bigwedge^2 T^*X^{1,0}))$ such that $\mu(\alpha) = -N_J(\alpha)$ for $\alpha \in \mathcal{A}^{0,1}$.

The tensor N_J is known as the Nijenhuis tensor and will play a key role in our discussion. Under the canonical identification $\text{Hom}(T^*X^{0,1}, \bigwedge^2 T^*X^{1,0}) \cong \bigwedge^2 T^*X^{1,0} \otimes TX^{0,1}$, we can view N_J as a skew-symmetric map

$$N_J : TX^{1,0} \times TX^{1,0} \rightarrow TX^{0,1} .$$

Lemma 2.28. *Under the identification above, the Nijenhuis tensor is given by*

$$N_J(X, Y) = ([X, Y])^{0,1} .$$

Proof. Let α be a $(0,1)$ -form and X, Y J -holomorphic vector fields. By the definition of μ and N_J , we have that $(N_J(\alpha))(X, Y) = -d\alpha(X, Y)$.

Now, we can expand the right-hand side using the usual formula $d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X, Y])$. The terms $\alpha(X)$ and $\alpha(Y)$ by bidegree reasons, and $\alpha([X, Y])$ only depends on the $(0,1)$ -part of the Lie bracket since α is a $(0,1)$ -form. $\square$

Exercise 2.29. *The usual definition of the Nijenhuis is*

$$\widetilde{N}_J(X, Y) = [X, Y] + J([JX, Y] + [X, JY]) - [JX, JY] .$$

Prove that the two definitions are equivalent (up to complexification and conjugation).

All in all, we have almost proved the following:

Proposition 2.30. *On an almost complex manifold, the following are equivalent:*

- (i) $\mu = 0$,
- (ii) *The subbundle $TX^{1,0}$ is involutive,*
- (iii) $\partial^2 = 0$.

Proof. The equivalence between (i) and (ii) follows from Lemma 2.28. Item (i) implies (iii) by Lemma 2.27. Thus, we only need to show that (iii) implies (ii).

It suffices to show that $\bar{\partial}f([X, Y]) = 0$ for a function f and $X, Y \in TX^{1,0}$. Now, we have

$$\begin{aligned} 0 &= \partial^2 f(X, Y) = (d\partial f)(X, Y) = X(\partial f(Y)) - Y(\partial f(X)) - \partial f([X, Y]) \\ &= X(df(Y)) - Y(df(X)) - \partial f([X, Y]) = df([X, Y]) - \partial f([X, Y]) \\ &= \bar{\partial}f([X, Y]) . \end{aligned} \quad \square$$

An almost complex structure is called integrable if any of the above conditions is satisfied, motivated by the following computation:

Lemma 2.31. *Let (X, J) be a complex manifold. Then $N_J \equiv 0$.*

Proof. Let $\{z_1, \dots, z_n\}$ be local holomorphic coordinates. Then $\{dz_1, \dots, dz_n\}$ is (pointwise) a basis for $T^*X^{1,0}$. In particular any $\alpha \in \mathcal{A}^{1,0}$ can be locally written as

$$\alpha = \sum_{k=1}^n f_k dz_k ,$$

In particular, we have

$$d\alpha = \sum_{j,k=1}^n \left(\frac{\partial f_k}{\partial z_j} dz_j + \frac{\partial f_k}{\partial \bar{z}_j} d\bar{z}_j \right) \wedge dz_k . \quad \square$$

So the vanishing of the Nijenhuis tensor is a necessary condition for (X, J) to be a complex manifold. In fact, it is also sufficient:

Theorem 2.32 (Newlander–Nirenberg). *An almost complex manifold (X, J) admits a compatible complex structure if and only if the almost complex structure J is integrable, i.e. $N_J \equiv 0$*

The proof of the Newlander–Nirenberg amounts to constructing local J -holomorphic coordinates. The details of the proof are relatively technical and involved; therefore, we will skip them. You can find a complete proof in [Dem12]

Therefore, one could define a complex manifold as a manifold equipped with an integrable almost complex structure.

Remark 2.33. In fact, one can take a more systematic approach to these questions from the point of view of G -structures. In that framework, the existence of an almost complex structure corresponds to a reduction of the frame bundle to a principal $\mathrm{GL}(n, \mathbb{C})$ -bundle, the vanishing of the Nijenhuis tensor corresponds to the structure being 1-integrable, and the Newlander–Nirenberg theorem says that there are no further obstructions from being 1-integrable to being integrable.

We will (hopefully) revisit the world of G -structures when we discuss the Kähler condition in Section 8.

A straightforward application of the Newlander–Nirenberg is the following:

Corollary 2.34. *Let $\iota : Z \hookrightarrow X$ be an almost complex submanifold of a complex manifold. Then Z is a complex submanifold of X*

2.3 Cohomologies in complex manifolds

As part of our discussion, we saw that (almost) complex manifolds carry natural operators that square to 0. In particular, this allows us to consider new cohomology theories for these operators.

Remark 2.35. The case of almost complex manifolds is not particularly amenable to having a good cohomology theory since the operator μ is of order 0, so cohomology groups will contain little interesting information. However, one can take this further to produce an interesting cohomology theory, but more elaborate tools are needed to realise this; see [CW21] for further details.

From now on, we restrict ourselves to the case of complex manifolds. Recall that, since $d^2 = 0$ and $d = \partial + \bar{\partial}$ on a complex manifold, we have $\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$. We can define four different cohomology theories on X :

Definition 2.36. Let (X, J) be a complex manifold.

- The *Dolbeault* cohomology

$$H_{\bar{\partial}}^{p,q}(X) = \frac{\ker(\bar{\partial} : \mathcal{A}^{p,q}(X) \rightarrow \mathcal{A}^{p,q+1}(X))}{\operatorname{im}(\bar{\partial} : \mathcal{A}^{p,q-1}(X) \rightarrow \mathcal{A}^{p,q}(X))}.$$

- The *de Rham* cohomology

$$H_{dR}^k(X) = \frac{\ker(d : \mathcal{A}^k(X) \rightarrow \mathcal{A}^{k+1}(X))}{\operatorname{im}(d : \mathcal{A}^{k-1}(X) \rightarrow \mathcal{A}^k(X))}.$$

item The *Bott–Chern* cohomology

$$H_{BC}^{p,q}(X) = \left(\frac{\ker \partial \cap \ker \bar{\partial}}{\operatorname{im} \partial \bar{\partial}} \right)^{p,q}$$

- The *Aeppli* cohomology

$$H_A^{p,q}(X) = \left(\frac{\ker \partial \bar{\partial}}{\operatorname{im} \partial + \operatorname{im} \bar{\partial}} \right)^{p,q}.$$

These are all well-defined, and there are canonical inclusion maps between the different cohomologies, induced by inclusion and projection:

$$\begin{array}{ccccc} & & H_{BC}^{p,q}(X) & & \\ & \swarrow & \downarrow & \searrow & \\ H_{\partial}^{p,q}(X) & & H_{dR}^{p+q}(X) & & H_{\bar{\partial}}^{p,q}(X) \\ & \searrow & \downarrow & \swarrow & \\ & & H_A^{p,q}(X) & & \end{array}$$

where $H_{\partial}^{p,q}(X)$ are defined analogously to the Dolbeault cohomology groups, and conjugation yields the isomorphisms $H_{\partial}^{p,q} \cong \overline{H_{\bar{\partial}}^{q,p}}$.

We conclude this section by computing the Dolbeault cohomology groups $H_{\bar{\partial}}^{p,q}$ on a polydisc $D_{\varepsilon} \subseteq \mathbb{C}^n$, for $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$, with $\varepsilon_i = \infty$ allowed. First, we need

Lemma 2.37 (Baby $\bar{\partial}$ -Poincaré Lemma). *Let $U \subseteq \mathbb{C}$ be an open set containing the closed ball $\overline{B_{\varepsilon}}$. For any $\alpha = f d\bar{z} \in \mathcal{A}^{0,1}(U)$, the function*

$$g = \frac{1}{2\pi i} \int_{B_{\varepsilon}} \frac{f(w)}{w - z} dw \wedge d\bar{w}$$

satisfies $\alpha = \bar{\partial}g$ on B_{ε} .

Proof. Let us prove that $\alpha = \bar{\partial}g$ in a neighbourhood V of $z_0 \in B_{\varepsilon}$. Take ψ a bump function such that $\psi|_V \equiv 1$ and $\operatorname{supp}(\psi) \subseteq B_{\varepsilon}$, and consider the decomposition $f = \psi f + (1 - \psi)f =: f_1 + f_2$, and the induced one for g . Let us check that g_1 is a well-defined smooth function. Since f_1 has

compact support, we can extend it to the entire complex plane, and by the change of coordinates $w = z + re^{i\phi}$, we have

$$\begin{aligned} \frac{1}{2\pi i} \int_{B_\varepsilon} \frac{f_1(w)}{w-z} dw \wedge d\bar{w} &= \frac{1}{2\pi i} \int_{\mathbb{C}} f(z + re^{i\phi}) \frac{(e^{i\phi} dr + ire^{i\phi} d\phi) \wedge (e^{-i\phi} dr - ire^{-i\phi} d\phi)}{re^{i\phi}} \\ &= \frac{1}{\pi} \int_{\mathbb{C}} f(z + re^{i\phi}) e^{-i\phi} d\phi \wedge dr, \end{aligned}$$

which is clearly smooth in B .

All that remains is to compute $\bar{\partial}g$. Since $\frac{1}{(w-z)}$ is holomorphic in the complement of V , it follows from differentiation under the integral sign that $\bar{\partial}g_2 = 0$. For g_1 , using the expression above, we have

$$\begin{aligned} \bar{\partial}g_1 &= \frac{1}{\pi} \bar{\partial} \int_{\mathbb{C}} f(z + re^{i\phi}) e^{-i\phi} d\phi \wedge dr \\ &= \frac{1}{\pi} \int_{\mathbb{C}} \left(\frac{\partial f}{\partial w} \frac{\partial(z + re^{i\phi})}{\partial \bar{z}} + \frac{\partial f}{\partial \bar{w}} \overline{\left(\frac{\partial(z + re^{i\phi})}{\partial z} \right)} \right) e^{-i\phi} d\phi \wedge dr \\ &= \frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial f}{\partial \bar{w}} e^{-i\phi} d\phi \wedge dr = \frac{1}{2\pi i} \int_B \frac{\partial f}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w-z} \\ &= f(z), \end{aligned}$$

where the second line follows from the chain rule from Lemma 1.3, we undid the change of variables in the third line, and the fourth line follows by the (general) Cauchy Integral Formula, Equation (3). $\square$

By induction on the dimension and bidegree, one shows

Lemma 2.38 ($\bar{\partial}$ -Poincaré lemma). *Let $U \subseteq \mathbb{C}^n$ be an open set containing the closed polydisc $\overline{D_\varepsilon}$. For $q > 0$, if $\alpha \in \mathcal{A}^{p,q}(U)$ is $\bar{\partial}$ -closed, there exists $\beta \in \mathcal{A}^{p,q-1}(D_\varepsilon)$ such that $\alpha = \bar{\partial}\beta$ on the polydisc.*

Proof. See [Huy05, Prop. 1.3.8]. $\square$

We can now prove the Dolbeault–Grothendieck lemma:

Proposition 2.39. *Let D_ε be a polydisc in $\mathbb{C}^n$. Then*

$$H_{\bar{\partial}}^{p,q}(D_\varepsilon) = \begin{cases} \text{holomorphic } (p\text{-forms}) & q = 0, \\ 0 & q > 0. \end{cases}$$

Proof. The idea is to exhaust the polydisc D_ε by a sequence of approximating polydiscs D_{ε_i} , and show that we can choose the approximating exact terms so that they do not change inside the smaller polydisc.

If $q > 1$, the difference $\beta_i - \beta_{i-1}$ will then be $\bar{\partial}$ -closed, so by the $\bar{\partial}$ -Poincaré lemma, we can choose γ_i such that $\bar{\partial}\gamma_i = \beta_i - \beta_{i-1}$. Take ψ a bump function supported on D_{ε_i} with $\psi|_{D_{\varepsilon_i}} = 1$ and set $\hat{\beta}_{i+1} = \beta_{i+1} + \bar{\partial}(\psi\gamma_i)$. The sequence $\hat{\beta}_i$ has the desired properties. The case $q = 1$ follows a similar idea, where now $\bar{\partial}\gamma$ is replaced by a suitable holomorphic polynomial. Full details can be found in [Huy05, Cor. 1.3.9]. $\square$

3 Sheaves and their cohomologies

We now introduce the language and techniques of sheaf theory. While we will not use them to their fullest extent, they are a convenient tool for presenting and proving some of our results, especially when considering cohomology and vector bundles. A more detailed discussion can be found in [Wel08] and references therein. For a more thorough and comprehensive discussion using derived functors, we refer the reader to [Har77, §3].

Definition 3.1. A *presheaf* $\mathcal{F}$ of abelian groups on a topological space X is given by:

- (i) For every open set $U \subseteq X$, an abelian group $\mathcal{F}(U)$
- (ii) For every inclusion $V \subseteq U$, a group morphism $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$ (restriction map)

such that $r_{UU} = \text{id}$ and $r_{VW} \circ r_{UV} = r_{UW}$ for $W \subseteq V \subseteq U$.

Definition 3.2. A presheaf is called a *sheaf* if for every family of sections $s_i \in \mathcal{F}(U_i)$, $i \in I$, with $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$. Equivalently, the sequence:

$$0 \rightarrow \mathcal{F}(U) \rightarrow \prod_i \mathcal{F}(U_i) \rightarrow \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is exact, where the second map is $(s_i) \mapsto (s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j})$.

We can now give a (perhaps) more intuitive definition of a stalk as a direct limit of a presheaf.

Definition 3.3. The *stalk* of a presheaf $\mathcal{F}$ at $x \in X$ is:

$$\mathcal{F}_x := \varinjlim_{x \in U} \mathcal{F}(U) = \bigcup_{x \in U} \mathcal{F}(U) / \sim$$

where $s_U \sim s_V$ if $s_U|_W = s_V|_W$ for some $x \in W \subseteq U \cap V$.

Associated with a presheaf, we have an associated topological space:

Definition 3.4. For a presheaf $\mathcal{F}$, define its *Étale* space:

$$\text{Ét}(\mathcal{F}) := \bigcup_{x \in X} \mathcal{F}_x \xrightarrow{p} X \quad \text{with} \quad p^{-1}(x) = \mathcal{F}_x$$

The sets $[U, s] = \{s_x \mid x \in U\}$ for U open and $s \in \mathcal{F}(U)$, form a basis for a topology on $\text{Ét}(\mathcal{F})$, and p is a local homeomorphism.

The *sheafification* $\mathcal{F}^+$ of a presheaf $\mathcal{F}$ is defined by:

$$\mathcal{F}^+(U) = \{s : U \rightarrow \text{Ét}(\mathcal{F}) \mid s \text{ is a continuous section}\}$$

There is a natural map $\mathcal{F}(U) \rightarrow \mathcal{F}^+(U)$ compatible with restrictions. If $\mathcal{F}$ is a sheaf, this map is an isomorphism.

An easy (but important) example is that of the constant presheaf and the locally constant sheaf:

Example 3.5. If $\mathcal{F}^{const}$ is the constant presheaf with $\mathcal{F}^{const}(U) = A$, then:

$$\acute{E}t(\mathcal{F}^{const}) = X \times A^{disc}, \quad (\mathcal{F}^{const})^+ = \underline{A}$$

Given a morphism of sheaves, we can study the associated kernel and image. First, we have

Lemma 3.6. Let $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves. Then the presheaf $\ker \varphi$ is a sheaf.

Proof. To prove that $\ker \varphi$ is a sheaf, we need to prove that, for U open and $\{U_i\}$ an open cover of U , we have

- (i) (Existence) if $s_i \in \ker \varphi(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there exists $s \in \ker \varphi(U)$ such that $s|_{U_i} = s_i$ for all i ;
- (ii) (Uniqueness) if $s \in \ker \varphi(U)$ and $s|_{U_i} = 0$, then $s = 0$.

To show (i), notice that the candidate s exists in $\mathcal{F}(U)$ since $\mathcal{F}$ is a sheaf. Thus, we only need to show that $s \in \ker \varphi(U)$. Indeed, $\varphi(s_i) = 0$ by hypothesis, and since $\mathcal{G}$ is also a sheaf, this glue together to show that $\varphi(s) = 0$, as needed. Uniqueness follows readily since $\mathcal{F}$ is a sheaf. $\square$

In general, however the presheaves $U \mapsto \operatorname{im} \varphi_U$ and $U \mapsto \operatorname{coker} \varphi_U$ are not sheaves. For instance, one may consider the image presheaf of the exponential map $\exp : \mathcal{O}_{\mathbb{C}} \rightarrow \mathcal{O}_{\mathbb{C}}^*$. Then, for an open set U , $\exp(U)$ is the ring of holomorphic functions on U with a well-defined logarithm. But taking $U_1 = \mathbb{C} \setminus \{x \geq 0\}$ and $U_2 = \mathbb{C} \setminus \{x \leq 0\}$ suffices to see that the image presheaf is not a sheaf, as there is no logarithm defined in $\mathbb{C} \setminus \{0\}$.

Definition 3.7. For a morphism $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ of sheaves, we define:

- The *image sheaf*: $\operatorname{im} \varphi := (U \mapsto \operatorname{im} \varphi_U)^+$
- The *cokernel sheaf*: $\operatorname{coker} \varphi := (U \mapsto \operatorname{coker} \varphi_U)^+$

A sequence $\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$ is called *exact* at $\mathcal{G}$ if $\ker \psi = \operatorname{im} \varphi$.

Similarly, we say the morphism φ is *injective* if $0 \rightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ is exact; and *surjective* if $\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \rightarrow 0$ is exact.

We have the following useful characterisation of exactness:

Lemma 3.8. The sequence $\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$ is exact iff $\mathcal{F}_x \xrightarrow{\varphi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x$ is exact for all $x \in X$.

Proof. Exercise. $\square$

The following sequences are examples of exact sequences:

$$\begin{aligned} 0 &\rightarrow \mathcal{O}_{\mathbb{C}} \xrightarrow{(z-p)\cdot} \mathcal{O}_{\mathbb{C}} \rightarrow S_{\mathbb{C}}(p) \rightarrow 0 \\ 0 &\rightarrow \underline{\mathbb{Z}} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \rightarrow 0 \\ 0 &\rightarrow \underline{\mathbb{C}} \rightarrow \mathcal{A}_{X,\mathbb{C}}^0 \xrightarrow{d} \mathcal{A}_{X,\mathbb{C}}^1 \rightarrow \dots \\ 0 &\rightarrow \Omega_X^p \rightarrow \mathcal{A}_{X,\mathbb{C}}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{A}_{X,\mathbb{C}}^{p,1} \rightarrow \dots \\ 0 &\rightarrow \mathcal{I}_p \rightarrow \mathcal{O}_X \rightarrow S_X(p) \rightarrow 0 \end{aligned}$$

where

- $\mathcal{O}_X$ is the sheaf of holomorphic functions on X ,
- $\mathcal{A}_{X,\mathbb{C}}^k$ (resp. $\mathcal{A}_{X,\mathbb{C}}^{p,q}$) is the sheaf of smooth sections of $\bigwedge^k T^*X$ (resp. $\bigwedge^{p,q} T^*X$),
- $\mathcal{I}_Y$ is the sheaf of vanishing holomorphic functions on a complex submanifold, $Y \subseteq X$

$$\mathcal{I}_Y(U) := \{f \in \mathcal{O}_X(U) \mid f|_Y = 0\}$$

- $S_X(0)$ is the skyscraper sheaf, defined as

$$S_X(p)(U) = \begin{cases} \mathbb{C} & \text{if } p \in U \\ 0 & \text{otherwise} \end{cases}.$$

, The reader is encouraged to go through these examples in detail and verify that they are exact sequences of sheaves, as they will appear repeatedly throughout the course.

Given a continuous map $f : X \rightarrow Y$ between topological spaces, we get induced maps on sheaves on them.

Definition 3.9. Let $f : X \rightarrow Y$ a continuous map, $\mathcal{F}$ a sheaf on X and $\mathcal{G}$ a sheaf on Y .

- The *direct image sheaf* of $\mathcal{F}$ is defined as $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$ for $U \subseteq Y$.
- The *inverse image sheaf* of $\mathcal{G}$ is defined as $f^{-1}\mathcal{G}(U) = \lim_{f(U) \subseteq V} \mathcal{G}(V)$, where the direct limit runs over all open subsets V of Y that contain $f(U)$.

One needs to check that the definitions are indeed well-posed, i.e. that the presheaves defined above are indeed sheaves; but we omit that.

The direct and inverse image sheaves satisfy some nice properties:

Lemma 3.10. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be continuous maps. Then,

- $g_* \circ f_* = (g \circ f)_*$ and $f^{-1} \circ g^{-1} = (g \circ f)^{-1}$,
- f^{-1} is exact (i.e. it preserves exactness),
- f_* and f^{-1} are adjoint to each other: $\text{Hom}(f^{-1}\mathcal{F}, \mathcal{G}) = \text{Hom}(\mathcal{F}, f_*\mathcal{G})$.

Lemma 3.11. Consider $\iota : Z \hookrightarrow X$ a continuous embedding, and $\mathcal{F}$ a sheaf on X . Let $\mathcal{F}|_Z = \iota^{-1}\mathcal{F}$. Then,

- if $Z = \{x\}$ is a point, $\mathcal{F}|_Z = \mathcal{F}_x$,
- if Z is closed, $\mathcal{F}(Z) = \mathcal{F}|_Z(Z)$, and
- if Z is open, $\mathcal{F}|_Z(V) = \mathcal{F}(Z \cap V)$.

We omit the proofs of these lemmas. Finally, for completeness, we introduce the following definitions

Definition 3.12. A *ringed space* is a pair $(X, \mathcal{R})$ where $\mathcal{R}$ is a sheaf of rings on X .

A *morphism* of ringed spaces $(X, \mathcal{R}) \rightarrow (Y, \mathcal{S})$ is a continuous map $f : X \rightarrow Y$ together with a morphism of sheaves of rings $f^{-1}\mathcal{S} \rightarrow \mathcal{R}$.

Definition 3.13. Let $(X, \mathcal{R})$ be a ringed space. A *sheaf of $\mathcal{R}$ -modules* is a sheaf of abelian groups $\mathcal{M}$ with a map $\mathcal{R} \times \mathcal{M} \rightarrow \mathcal{M}$ such that $\mathcal{M}(U)$ is an $\mathcal{R}(U)$ -module for all open U .

Examples of ringed spaces are smooth manifolds, with $\mathcal{R} = \mathcal{C}_X^\infty (= \mathcal{A}_{X, \mathbb{R}}^0)$, and complex manifolds, with $\mathcal{R} = \mathcal{O}_X$. Examples of $\mathcal{R}$ -modules are discussed in the exercises.

3.1 Sheaf cohomology

Let us now discuss the issue of exactness (or rather its failure). We saw (or rather left as an exercise) that taking stalks is an exact operation. More generally, we have

Lemma 3.14. *Let*

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$$

be a short exact sequence of sheaves. Then, for any U , we have

$$0 \rightarrow \mathcal{F}(U) \rightarrow \mathcal{G}(U) \rightarrow \mathcal{H}(U)$$

Proof.

□

In general, we lose exactness on the right, as exemplified by the fact that the exponential map $\exp : \mathcal{O}_{\mathbb{C}} \rightarrow \mathcal{O}_{\mathbb{C}}^*$ is not surjective when evaluated over $U = \mathbb{C} \setminus \{0\}$.

Cohomology is then introduced as a measure of failure for right-exactness. The correct way to understand sheaf cohomology is via the theory of derived functors, which is unfortunately beyond the scope of this course. Instead, we will present an ad-hoc construction for it.

Definition 3.15. A sheaf $\mathcal{I}$ is *injective* if for any injection $\mathcal{A} \hookrightarrow \mathcal{B}$ and map $\mathcal{A} \rightarrow \mathcal{I}$, there exists a map $\mathcal{B} \rightarrow \mathcal{I}$ making the diagram commute.

$$\begin{array}{ccc} A & \hookrightarrow & B \\ & \searrow & \downarrow \\ & & I \end{array}$$

Definition 3.16. A *complex of sheaves* is a sequence:

$$\dots \rightarrow \mathcal{F}^{i-1} \xrightarrow{d} \mathcal{F}^i \xrightarrow{d} \mathcal{F}^{i+1} \rightarrow \dots$$

A *resolution* of a sheaf $\mathcal{F}$ is a complex $\mathcal{F}^\bullet$ with a map $\mathcal{F} \hookrightarrow \mathcal{F}^0$ that is exact. An *injective resolution* is a resolution where all $\mathcal{I}^i$ are injective.

Definition 3.17. The *sheaf cohomology* is defined as:

$$H^i(X, \mathcal{F}) := H^i(\Gamma(X, \mathcal{I}^\bullet))$$

for an injective resolution $\mathcal{F} \rightarrow \mathcal{I}^\bullet$.

Notice that, in particular $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F}) = \mathcal{F}(X)$. A priori, this definition is subject to the existence of injective resolutions and a choice thereof. Fortunately, we have:

Proposition 3.18.

- (i) Every sheaf $\mathcal{F}$ admits an injective resolution. (The category of sheaves has enough injectives.)
- (ii) For a morphism of sheaves $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ and injective resolutions $\mathcal{I}^\bullet$ and $\mathcal{J}^\bullet$ of $\mathcal{F}$ and $\mathcal{G}$, there exist $\varphi^k : \mathcal{I}^k \rightarrow \mathcal{J}^k$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{I}^0 & \longrightarrow & \mathcal{I}^1 & \longrightarrow & \mathcal{I}^2 & \longrightarrow & \dots \\ & & \downarrow \varphi & & \downarrow \varphi^0 & & \downarrow \varphi^1 & & \downarrow \varphi^2 & & \\ 0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{J}^0 & \longrightarrow & \mathcal{J}^1 & \longrightarrow & \mathcal{J}^2 & \longrightarrow & \dots \end{array}$$

commutes. Moreover, any choice of maps $\{\varphi^k\}$ induces the same maps on cohomology.

- (iii) Injective sheaves are flabby, i.e. the map $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$ is surjective for any $V \subseteq U$ open.
- (iv) If

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$$

is exact and $\mathcal{F}$ is flabby, then

$$0 \rightarrow \mathcal{F}(U) \rightarrow \mathcal{G}(U) \rightarrow \mathcal{H}(U) \rightarrow 0$$

for all open subsets U .

In particular, this implies that the sheaf cohomology groups are well-defined, and we have

Theorem 3.19. Consider the short exact sequence of sheaves

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0.$$

Then there exists a long exact sequence of cohomology:

$$0 \rightarrow H^0(X, \mathcal{F}) \rightarrow H^0(X, \mathcal{G}) \rightarrow H^0(X, \mathcal{H}) \rightarrow H^1(X, \mathcal{F}) \rightarrow H^1(X, \mathcal{G}) \rightarrow H^1(X, \mathcal{H}) \rightarrow H^2(X, \mathcal{F}) \rightarrow \dots$$

is exact

Proof. Use the fact that the injective resolution is flabby, along with the snake lemma/ diagram chasing, to construct the connecting morphisms. $\square$

Whilst injective sheaves and injective resolutions are convenient to define sheaf cohomology, they tend to be quite cumbersome and hard to construct in explicit situations. Instead, it is more convenient to work with acyclic sheaves and resolutions

Definition 3.20. A sheaf $\mathcal{A}$ is *acyclic* if $H^i(X, \mathcal{A}) = 0$ for $i > 0$. An *acyclic resolution* is a resolution $\mathcal{A}^\bullet$ by acyclic sheaves $\mathcal{A}^i$.

The following result captures the convenience of working with acyclic resolution.

Theorem 3.21. *Let $\mathcal{A}^\bullet$ be an acyclic resolution of $\mathcal{F}$, then:*

$$H^i(X, \mathcal{F}) = H^i(\Gamma(X, \mathcal{A}^\bullet))$$

Proof. Split the resolution into short exact sequences:

$$0 \rightarrow \mathcal{K}^i \rightarrow \mathcal{A}^i \rightarrow \mathcal{K}^{i+1} \rightarrow 0$$

with $\mathcal{K}^i := \ker(A^i \rightarrow A^{i+1}) \cong \operatorname{im}(A^{i-1} \rightarrow A^i)$. The long exact sequence of cohomology yields the desired result. $\square$

Exercise 3.22. *Write down the missing details of the proof above.*

We now claim a fact that will be of great importance, but we do not have the time to prove it:

Theorem 3.23. *All sheaves of $\mathcal{A}_{\mathbb{R}}$ -modules are acyclic.*

The proof of the theorem relies on constructing a particular type of acyclic sheaves called *soft*, via a partition of unity on X . This dependence on the existence of a partition of unity is key in the construction.

As a corollary of this fact, we have

Corollary 3.24. *Let X be a smooth manifold. Then*

$$H_{dR}^k(X, \mathbb{R}) \cong H^k(X, \underline{\mathbb{R}}).$$

Similarly, on a complex manifold, we have

$$H_{\bar{\partial}}^{p,q}(X) \cong H^q(X, \Omega^p)$$

Proof. The smooth Poincaré lemma implies that the locally constant sheaf $\underline{\mathbb{R}}$ admits the acyclic resolution

$$\mathcal{A}_{X,\mathbb{R}}^\bullet := 0 \rightarrow \mathcal{A}_{X,\mathbb{R}}^0 \xrightarrow{d} \mathcal{A}_{X,\mathbb{R}}^1 \xrightarrow{d} \mathcal{A}_{X,\mathbb{R}}^2 \xrightarrow{d} \dots$$

Similarly, the $\bar{\partial}$ -Poincaré lemma implies that sheaf of holomorphic p -forms admits the acyclic resolution

$$\mathcal{A}_{X,\mathbb{R}}^{p,\bullet} := 0 \rightarrow \mathcal{A}_{X,\mathbb{R}}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{A}_{X,\mathbb{R}}^{p,1} \xrightarrow{\bar{\partial}} \mathcal{A}_{X,\mathbb{R}}^{p,2} \xrightarrow{\bar{\partial}} \dots$$

$\square$

3.2 Čech cohomology

We now introduce another, more combinatorial, cohomology theory for sheaves. Whilst it is more "hands-on" and computationally easy to work with, one does not have all the good properties of sheaf cohomology "on the nose".

Definition 3.25. Let $\mathcal{F}$ be a sheaf on X and $\mathcal{U} = \{U_i\}_{i \in I}$ an open cover. For each $\sigma = (i_0, \dots, i_q) \in I^{q+1}$, consider $U_\sigma = U_{i_0} \cap \dots \cap U_{i_q}$ and $\iota_\sigma : U_\sigma \hookrightarrow X$ the inclusion.

(i) The *sheaf of Čech chains* with respect to the cover $\mathcal{U}$ is:

$$\mathcal{C}^q(\mathcal{U}, \mathcal{F}) = \prod_{\sigma \in I^{q+1}} (\iota_\sigma)_* (\iota_\sigma)^{-1} \mathcal{F}$$

(ii) The *Čech boundary operator* is:

$$\begin{aligned} \delta : \mathcal{C}^q(\mathcal{U}, \mathcal{F}) &\rightarrow \mathcal{C}^{q+1}(\mathcal{U}, \mathcal{F}) \\ (s_\sigma)_\sigma &\mapsto \sum_{k=0}^{q+1} (-1)^k (s_{i_0, \dots, \check{i}_k, \dots, i_{q+1}})|_{U_{i_0, \dots, i_{q+1}}} \end{aligned}$$

A (tedious) computation shows that $\delta^2 = 0$, so $(\mathcal{C}^q(\mathcal{U}, \mathcal{F}), \delta)$ is a complex of sheaves. In particular, we can define the (relative) Čech cohomology groups:

$$\check{H}^q(\mathcal{U}, \mathcal{F}) := \frac{\ker \left(\mathcal{C}^q(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} \mathcal{C}^{q+1}(\mathcal{U}, \mathcal{F}) \right)}{\operatorname{im} \left(\mathcal{C}^{q-1}(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} \mathcal{C}^q(\mathcal{U}, \mathcal{F}) \right)}.$$

In degree zero, we have

$$C^0(\mathcal{U}, \mathcal{F}) = \prod_{U_i} \mathcal{F}(U_i) \xrightarrow{\delta} \prod_{U_i \cap U_j} \mathcal{F}(U_i \cap U_j) = C^1(\mathcal{U}, \mathcal{F})$$

with $\delta(s)_{ij} = s_j|_{U_i \cap U_j} - s_i|_{U_i \cap U_j}$. Since $\mathcal{F}$ is a sheaf, $\check{H}^0(\mathcal{U}, \mathcal{F}) = \ker \delta = H^0(X, \mathcal{F})$. However, the higher cohomology groups will depend on the chosen cover. To remedy this, we define

Definition 3.26. Let X be a topological space and $\mathcal{F}$ a sheaf. We define the *Čech cohomology groups* as

$$\check{H}^q(X, \mathcal{F}) = \lim_{\mathcal{U} \text{ cover}} \check{H}^q(\mathcal{U}, \mathcal{F}),$$

where the direct limit is taken over finer and finer covers.

The result that ties up all the discussion is a celebrated result due to Leray:

Theorem 3.27 (Leray's theorem). : *Let X be a smooth manifold. There is an isomorphism:*

$$H^q(X, \mathcal{F}) \cong \check{H}^q(X, \mathcal{F})$$

The main idea is to consider a good cover of X , that is, an open cover in which all open sets and all non-empty intersections of finitely many of them are contractible, and then choosing a partition of unity subordinate to this open cover.

I have been particularly vague and stated many (deep and hard) results at face value, which the reader should be pretty unhappy about (I know I am). Unfortunately, I find it the lesser of all evils, as proceeding in our discussion without the tools of sheaf theory and its cohomologies would prove nearly impossible. However, establishing and discussing all the material summarised in this section in detail could take an entire course on its own.

4 Meromorphic functions and Siegel's theorem

Let us put together some of the results from the previous sections. In Theorem 1.24 we saw that the stalk $\mathcal{O}_{X,x}$ (equivalently $\mathcal{O}_{\mathbb{C}^n,0}$) is an integral domain since it is a UFD, as proved in Theorem 1.24. So one may consider the corresponding field of fractions $\mathcal{K}_{X,x} := \text{Quot}(\mathcal{O}_{X,x})$. Consider the following space

$$\acute{\text{Et}}(\mathcal{K}_X) := \bigcup_{x \in X} \mathcal{K}_x ,$$

with the topology induced by that of the étale space of $\mathcal{O}_X$, and define the following sheaf:

Definition 4.1. The *sheaf of meromorphic functions* on a complex manifold X is defined as

$$\mathcal{K}_X(U) = \left\{ s : U \rightarrow \acute{\text{Et}}(\mathcal{K}_X) \mid s \text{ is continuous and } p \circ s = \text{id}_U \right\} ,$$

with $p : \acute{\text{Et}}(\mathcal{K}_X) \rightarrow X$ the obvious projection. A *meromorphic function* is a section of this sheaf.

Note that we have chosen very suggestive notation from the start, and we are treating $\acute{\text{Et}}(\mathcal{K}_X)$ as an étale space, and constructed the sheaf out of it as we did for the sheafification of a presheaf.

This procedure is quite general and does not use any intrinsic properties of holomorphic functions. Indeed, this can be applied to any ringed space $(X, \mathcal{R})$ as long as the stalks of $\mathcal{R}$ are integral domains. The resulting construction is called the sheaf of rational functions.

Remark 4.2. Note that one might want to abuse notation and write $\mathcal{K}_X = \text{Quot}(\mathcal{O}_X)$. However, $\text{Quot}(\mathcal{O}_X(U))$ makes no sense for any open U that is not connected, since $\mathcal{O}_X(U)$ will not be an integral domain.

Let us study the sheaf of meromorphic functions. First, notice that if X is connected, $\mathcal{K}_X(X)$ is a field, and we have an injective map of sheaves $0 \rightarrow \mathcal{O}_X \xrightarrow{\iota} \mathcal{K}_X$ with $\iota(f) = \frac{f}{1}$.

Since meromorphic functions are continuous sections $s : X \rightarrow \acute{\text{Et}}(\mathcal{K}_X)$, we have the following characterization:

Lemma 4.3. *Let $f \in \mathcal{K}_X(U)$. Then, for every $x \in U$ there exists an open neighbourhood V and holomorphic functions $g, h \in \mathcal{O}_X(V)$ such that the stalks g_y and h_y are coprime and $f_y = \frac{g_y}{h_y}$ for all $y \in V$. Moreover, g and h are unique, up to units in $\mathcal{O}_X(V)$.*

Proof. Combine the definition of meromorphic functions using the topology of $\acute{\text{Et}}(\mathcal{K}_X)$ (and thus that of $\acute{\text{Et}}(\mathcal{O}_X)$) with the fact that $\mathcal{O}_{X,x}$ is a UFD and Lemma 1.25. $\square$

In particular, for any meromorphic function $f \in \mathcal{K}_X(U)$, we can define the following two analytic sets:

$$Z(f) := \left\{ x \in U \mid f_x = \frac{g_x}{h_x}, \quad g(x) = 0 \right\} \quad P(f) := \left\{ x \in U \mid f_x = \frac{g_x}{h_x}, \quad h(x) = 0 \right\}$$

Notice that when f is actually holomorphic, the definition of $Z(f)$ agrees with our previous definition $Z(f) = f^{-1}(0)$. In fact, a moment's thought suffices to notice that the sets $Z(f)$ and $P(f)$ are disjoint, and that $f|_{U \setminus P(f)}$ is holomorphic.

We know that, on a compact connected complex manifold, the ring of holomorphic functions is always just $\mathbb{C}$. Let us now study how "big" the field of meromorphic functions can be. Recall from your algebra course

Definition 4.4. Let K be a field and L a field extension over K . We say that a collection $\{l_1, \dots, l_k\} \subseteq L$ is transcendently independent if $\phi(l_1, \dots, l_k) \neq 0$ for all polynomials $\phi \in K[x_1, \dots, x_k]$.

The *transcendence degree* of the extension $L|K$ is defined as

$$\text{tr. deg}_K(L) = \sup_k \{k \mid \exists l_1, \dots, l_k \text{ transcendently independent over } K\}$$

We have

Theorem 4.5 (Siegel's theorem). *Let X be a compact connected complex manifold of dimension n . Then $\text{tr. deg}_{\mathbb{C}}(\mathcal{K}_X(X)) \leq n$.*

Before we can prove this, we need the following lemma:

Lemma 4.6 (Schwarz lemma). *Let $\varepsilon > 0$ and $f : \overline{B_\varepsilon(0)} \rightarrow \mathbb{C}$ a holomorphic function with a zero of order k at 0. Then*

$$|f(z)| \leq C \left(\frac{|z|}{\varepsilon} \right)^k,$$

where $C = \sup_{\partial D_\varepsilon(0)} |f|$.

Proof. Fix $0 \neq z \in D_\varepsilon(0)$ and consider the function

$$F_x : B_\varepsilon(0) \rightarrow \mathbb{C} \\ w \mapsto w^{-k} f\left(w \frac{z}{|z|}\right),$$

which is holomorphic since f has a zero of order k . On $\partial B_\varepsilon(0)$, we have $|F_x| \leq C\varepsilon^{-k}$. By the maximum principle, the same bound holds for all $w \in B_\varepsilon(0)$. Thus, taking $w = |z|$, we have

$$|F_x(|z|)| = |z|^{-k} |f(z)| \leq C\varepsilon^{-k}. \quad \square$$

Proof of Siegel's theorem. The goal is to prove that for all $f_1, \dots, f_{n+1} \in \mathcal{K}_X(X)$, there exists a polynomial $P \in K[x_1, \dots, x_{n+1}]$ such that $P(f_1, \dots, f_{n+1}) = 0$.

- Step 0: Let $z \in X$, so there exists an open neighbourhood U and $g_1, \dots, g_{n+1}, h_1, \dots, h_{n+1} \in \mathcal{O}(U)$ such that $f_i|_U = \frac{g_i}{h_i}$ for all $i \in \{1, \dots, n+1\}$. Moreover, we can assume that g_i and h_i are coprime by taking a smaller neighbourhood if necessary, by Lemma 1.25.

Take coordinate charts around z , and consider $V_x \subseteq U$ the image of the ball of radius $1/2$ under the chosen chart. Since X is compact, we can find points $z_1, \dots, z_N$ such that the family $\{V_k\}_k$ is an open cover of X .

For two neighbourhoods, U_k and U_l , denote $\phi_{kl,i} = \frac{g_{k,i}}{g_{l,i}} \in \mathcal{O}^*(U_k \cap U_l)$, and set

$$C = \max_{k,l} \sup_{z \in \overline{V_k} \cap \overline{V_l}} \left| \prod_{i=1}^{n+1} \phi_{kl,i}(z) \right|.$$

Notice that the relation $\phi_{kl,i}(z)\phi_{lk,i}(z) = 1$ implies $C \geq 1$.

- Step 1: There exists a polynomial P of degree m such that

$$G_k = P(f_1, \dots, f_{n+1}) \left(\prod h_{i,k} \right)^m$$

is holomorphic on U_k and vanishes at z_k with order M for all $k \in \{1, \dots, N\}$.

Indeed, G_k is clearly holomorphic. The condition that G_k vanishes at order M is equivalent to $\partial^\alpha G_k = 0$, where $\partial^\alpha := \frac{\partial^\alpha}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}}$ is the differential operator, and $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multindex of size M .

The collection of operators ∂^α spans a space of dimension $\binom{M-1+n}{n}$. Since the space of polynomials of degree m has dimension $\binom{m+n+1}{m}$, it suffices to choose m large enough so

$$\binom{m+n+1}{m} > N \binom{M-1+n}{n}. \quad (6)$$

- Step 2: By the Schwarz Lemma 4.6, we have

$$|G_k(z)| \leq \frac{C}{2^M}$$

for $z \in V_k$ and $D = \max_k \sup_{z \in \overline{V_k}} |G_k(z)|$. The goal is to show that $D = 0$ for an appropriate choice of m and M .

- Step 3: Let $z \in U_k$ such that $|G_k(z)| = D$. Thus, for some $l \in \{1, \dots, N\}$ $z \in V_l$ and so

$$D = |G_k(z)| = |G_l(z)| |\phi_{kl}^m(z)| \leq \frac{C^m}{2^M} D.$$

Thus, the constant D will vanish whenever

$$m \log_2(C) < M. \quad (7)$$

- Step 4: For suitably chosen m and M , the conditions (6) and (7) can be satisfied simultaneously, so $D \equiv 0$.

Indeed, since $\binom{m+n+1}{m}$ is a polynomial of degree $n+1$ in m and $\binom{M-1+n}{M-1}$ is a polynomial of degree n in M . Notice that this is the crucial step for which we need to take the polynomial in $n+1$ variables.

□

In view of Siegel's result, we see that the following definition makes sense:

Definition 4.7. The *algebraic dimension* of a compact connected complex manifold X is

$$a(X) := \text{tr. deg}_{\mathbb{C}} (\mathcal{K}_X(X)).$$

As a first computation of the algebraic dimension, we have:

Proposition 4.8. For all $n \in \mathbb{N}$, we have $a(\mathbb{CP}^n) = n$.

Proof. By Siegel's theorem, it suffices to show that $a(\mathbb{CP}^n) \geq n$. Let $[Z_0 : \dots : Z_n]$ denote homogeneous coordinates on $\mathbb{CP}^n$ and $\mathbb{C}(\xi_1, \dots, \xi_n)$ the field of rational functions. The map

$$\begin{aligned} \Phi : \mathbb{C}(\xi_1, \dots, \xi_n) &\rightarrow \mathcal{K}_{\mathbb{CP}^n}(\mathbb{CP}^n) \\ f(\xi_1, \dots, \xi_n) &\mapsto f\left(\frac{Z_1}{Z_0}, \dots, \frac{Z_n}{Z_0}\right) \end{aligned}$$

is well-defined, so $\mathbb{C}(\xi_1, \dots, \xi_n) \subseteq \mathcal{K}_{\mathbb{CP}^n}(\mathbb{CP}^n)$. $\square$

This proves that the field extension $\mathcal{K}_{\mathbb{CP}^n}(\mathbb{CP}^n) | \mathbb{C}(\xi_1, \dots, \xi_n)$ is algebraic. It is not hard to prove that, in fact, $\mathbb{C}(\xi_1, \dots, \xi_n) \cong \mathcal{K}_{\mathbb{CP}^n}$, but we leave it as an exercise to the reader.

5 Holomorphic bundles and Kodaira dimension

Recall the definition of smooth real (resp. complex) vector bundles:

Definition 5.1. A real (resp. complex) *vector bundle* of rank r over a manifold X is a smooth manifold E together with a smooth projection $\pi : E \rightarrow X$ such that there exists an open cover $\{U_i\}$ of X and diffeomorphisms $\varphi_i : \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{R}^r$ (resp. $\mathbb{C}^r$) such that:

- (i) $\pi = \text{pr}_1 \circ \varphi_i$ on $\pi^{-1}(U_i)$, where pr_1 denotes the projection to the first factor.
- (ii) On $U_i \cap U_j$, the transition functions $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : (U_i \cap U_j) \times \mathbb{R}^r \rightarrow (U_i \cap U_j) \times \mathbb{R}^r$ are of the form $(x, v) \mapsto (x, g_{ij}(x)v)$ where $g_{ij} \in \mathcal{C}^\infty(U_i \cap U_j, \text{GL}(r, \mathbb{R}))$

Therefore, one makes the analogue definition for the holomorphic case:

Definition 5.2. A *holomorphic vector bundle* of rank r on a complex manifold X is a complex manifold E together with a holomorphic projection $\pi : E \rightarrow X$ such that there exists an open cover $\{U_i\}$ of X and biholomorphic maps $\varphi_i : \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{C}^r$ such that:

- (i) $\pi = \text{pr}_1 \circ \varphi_i$ on $\pi^{-1}(U_i)$
- (ii) On $U_i \cap U_j$, the transition functions $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : (U_i \cap U_j) \times \mathbb{C}^r \rightarrow (U_i \cap U_j) \times \mathbb{C}^r$ are of the form $(x, v) \mapsto (x, g_{ij}(x)v)$ where $g_{ij} : U_i \cap U_j \rightarrow \text{GL}_r(\mathbb{C})$ are holomorphic.

The first obvious example of a holomorphic vector bundle is the holomorphic tangent bundle of a complex manifold:

Lemma 5.3. *For X a complex manifold, the bundle $T^{1,0}X \subseteq TX \otimes \mathbb{C}$ is holomorphic.*

Proof. Let $\{z_1, \dots, z_n\}$ be local coordinates on the complex manifold. Then $\left\{ \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n} \right\}$ are a local basis of $T^{1,0}X$. The holomorphicity condition on a change of basis between trivialisations is precisely the condition that X is a complex manifold. $\square$

As in the case of smooth vector bundles, any natural construction (in a category theory sense) of vector spaces gives rise to natural constructions of holomorphic vector bundles. In particular, we have:

Lemma 5.4. *Let E, F be holomorphic vector bundles. Then the following vector bundles are holomorphic:*

(i) $E \oplus F$,

(iii) E^* , the dual of E ,

(ii) $E \otimes F$,

(iv) $\bigwedge^k E$ for all $k > 0$,

Moreover, let $\varphi : E \rightarrow F$ a bundle morphism. Then the bundles $\ker \varphi$ and $\operatorname{coker} \varphi$ are holomorphic.

Vector bundles are classified by the appropriate (Čech) cohomology group:

Proposition 5.5. *Up to isomorphism, we have the following correspondences:*

- real vector bundles of rank $r \xleftarrow{1:1} \check{H}^1(X, \operatorname{GL}(r, \mathcal{C}^\infty(X, \mathbb{R})))$,
- complex vector bundles of rank $r \xleftarrow{1:1} \check{H}^1(X, \operatorname{GL}(r, \mathcal{C}^\infty(X, \mathbb{C})))$
- holomorphic vector bundles of rank $r \xleftarrow{1:1} \check{H}^1(X, \operatorname{GL}(r, \mathcal{O}_X))$,

where $\operatorname{GL}(r, \mathcal{F})$ is the sheaf of invertible rank k matrices with coefficients in the sheaf $\mathcal{F}$.

Proof. Exercise. □

Understanding and computing the groups $\check{H}^1(X, \operatorname{GL}(r, \mathcal{A}))$ is very hard, and there are no general results, except for the case $r = 1$, that we will revisit shortly.

To conclude this introduction, we introduce a generalisation of the Dolbeault operator $\bar{\partial}$ to holomorphic bundles $\bar{\partial}_E$.

Proposition 5.6. *Let $E \rightarrow X$ be a holomorphic bundle, and let $\mathcal{A}_X^{p,q}(E)$ the space of smooth (p, q) -forms with values in E . There exists a $\mathbb{C}$ -linear operator $\bar{\partial}_E : \mathcal{A}^{p,q}(E) \rightarrow \mathcal{A}^{p,q+1}(E)$ such that*

(i) *it satisfies the Leibniz rule, i.e. $\bar{\partial}_E(\alpha \wedge s) = \bar{\partial}\alpha \wedge s + (-1)^{p+q}\alpha \wedge \bar{\partial}_E s$ for $\alpha \in \mathcal{A}_X^{p,q}$ and $s \in \mathcal{A}_X^{p',q'}(E)$, and*

(ii) *it squares to zero, $\bar{\partial}_E^2 = 0$.*

Proof. Clearly, the operator $\bar{\partial}_E$ is local. Let $\{s_1, \dots, s_k\}$ be a local trivialisation of E , so any $s \in \mathcal{A}_X^{p,q}(E)$ is given by $\alpha_i \in \mathcal{A}_X^{p,q}$, with $s = \sum_{i=1}^k \alpha_i \otimes s_i$. We define

$$\bar{\partial}_E s := \sum_{i=1}^k \bar{\partial}(\alpha_i) \otimes s_i.$$

We need to check that this is well-defined. For another local trivialisation $\{t_1, \dots, t_k\}$, there exists a matrix $A = (\psi_{ij}) \in \operatorname{GL}(k, \mathcal{O}_X^*)$ such that $s_i = \sum_j \psi_{ji} \otimes t_j$. Thus,

$$\begin{aligned} \bar{\partial}_E s &= \sum_{i=1}^k \bar{\partial}(\alpha_i) \otimes s_i = \sum_{i,j=1}^k \bar{\partial}(\alpha_i) \otimes (\psi_{ji} \otimes t_j) \\ &= \sum_{j=1}^k \bar{\partial} \left(\sum_{i=1}^k \alpha_i \psi_{ji} \right) \otimes t_j = \bar{\partial}_E s, \end{aligned}$$

where we crucially used that E is holomorphic, so the transition functions are holomorphic, i.e. $\bar{\partial}\psi_{ij} = 0$. □

Conversely, we have

Theorem 5.7. *Let $E \rightarrow X$ be a complex vector bundle carrying an operator $\bar{\partial}_E$ satisfying the conditions above. Then E carries a natural holomorphic structure.*

The idea of the proof is that the integrability condition $\bar{\partial}_E^2 = 0$ acts like a "vanishing" Nijenhuis tensor, so one can adapt the Newlander-Nirenberg theorem (in fact Frobenius' theorem is enough) to produce the holomorphic local trivialising sections. We refer the reader to [Mor07, Thm. 9.2] for a direct proof using the N-N theorem, and to [DK90, Sec. 2.2] for a more general discussion. The integrability condition $\bar{\partial}_E^2 = 0$ allows us to consider a version of Dolbeault cohomology for vector bundles:

$$H_X^{p,q}(E) := \frac{\ker(\bar{\partial}_E : \mathcal{A}_X^{p,q}(E) \rightarrow \mathcal{A}_X^{p,q+1}(E))}{\operatorname{im}(\bar{\partial}_E : \mathcal{A}_X^{p,q-1}(E) \rightarrow \mathcal{A}_X^{p,q}(E))}.$$

Again, the $\bar{\partial}$ -Poincaré lemma implies that the complex $0 \rightarrow \mathcal{A}_X^{p,0}(E) \rightarrow \mathcal{A}_X^{p,1}(E) \rightarrow \dots$ is an acyclic resolution of $E \otimes \Omega_X^p$, so

Corollary 5.8. *We have $H_X^{p,q}(E) \cong H^q(X, E \otimes \Omega_X^p)$.*

5.1 Holomorphic line bundles

Let us now focus on studying line bundles. First, as we anticipated earlier, we have

Lemma 5.9.

- (i) *Complex line bundles over X are in one-to-one correspondence with elements of $H^2(X, \mathbb{Z})$.*
- (ii) *Real line bundles over X are in one-to-one correspondence with elements of $H^1(X, \mathbb{Z}/2\mathbb{Z})$.*

Proof. Consider the exponential sequence

$$0 \rightarrow \mathbb{Z} \xrightarrow{2\pi i} \mathcal{A}_{X,\mathbb{C}} \xrightarrow{\exp} \mathcal{A}_{X,\mathbb{C}}^* \rightarrow 0.$$

We have a long exact sequence of cohomology

$$\dots \rightarrow H^1(X, \mathcal{A}_{X,\mathbb{C}}) \rightarrow H^1(X, \mathcal{A}_{X,\mathbb{C}}^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathcal{A}_{X,\mathbb{C}}) \rightarrow \dots \quad (8)$$

Since $\mathcal{A}_{\mathbb{C}}$ is acyclic, the map $c_1 : H^1(X, \mathcal{A}_{\mathbb{C}}^*) \rightarrow H^2(X, \mathbb{Z})$ is a bijection. Similarly, for the real line bundle case, one considers the short exact sequence

$$0 \rightarrow \mathcal{A}_{X,\mathbb{R}} \xrightarrow{\exp} \mathcal{A}_{X,\mathbb{R}}^* \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0. \quad \square$$

Whilst $H^1(X, \operatorname{GL}(r, \mathcal{F}))$ does not carry any additional structure for $r > 1$, $H^1(X, \operatorname{GL}(1, \mathcal{F})) \cong H^1(X, \mathcal{F}^*)$ always carries the additional structure of an abelian group:

Lemma 5.10. *The set $H^1(X, \mathcal{F}^*)$ carries the structure of an abelian group, where the tensor product induces the group operation, and inverses are given by dualisation [$L^{-1} := L^*$].*

Proof. Immediate. □

Corollary 5.11. *The maps*

$$c_1 : H^1(X, \mathcal{A}_{\mathbb{C}}^*) \rightarrow H^2(X, \mathbb{Z}) \qquad w_1 : H^1(X, \mathcal{A}_{\mathbb{R}}^*) \rightarrow H^1(X, \mathbb{Z}/2\mathbb{Z})$$

are group morphisms.

Let us now focus on the case of holomorphic line bundles:

Definition 5.12. The group of isomorphism classes of line bundles is called the *Picard group*:

$$\text{Pic}(X) := H^1(X, \mathcal{O}_X^*).$$

Again, by using the exponential short exact sequence, we have:

Proposition 5.13.

- (i) *A complex line bundle L admits a holomorphic structure if and only if $c_1(L)$ maps to zero in $H^2(X, \mathcal{O}_X)$.*
- (ii) *The set of (non-isomorphic) holomorphic structures on a holomorphic line bundle is parametrised by $H^1(X, \mathcal{O}_X)/\text{im}(H^1(X, \mathbb{Z}))$.*

Proof. Comparing the smooth and holomorphic exponential sequences, we have:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_X^* \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{A}_{\mathbb{C}} & \longrightarrow & \mathcal{A}_{\mathbb{C}}^* \longrightarrow 0 \end{array}$$

The claim follows from the induced map of long exact sequences. □

In particular, this discussion shows that, over a complex manifold with $H^2(X, \mathcal{O}_X) = 0$, any complex line bundle admits a holomorphic structure. As in the case of almost complex manifolds, this is not true for higher rank complex bundles, as we shall see.

Let us now introduce the tautological line bundle of the complex projective space $\mathbb{CP}^n$:

Proposition 5.14. *The tautological line bundle $\mathcal{O}(-1)$ on $\mathbb{P}^n$ is the line incidence variety:*

$$\mathcal{O}(-1) = \{(l, z) \mid z \in l\} \subseteq \mathbb{P}^n \times \mathbb{C}^{n+1}$$

with projection $\pi : \mathcal{O}(-1) \rightarrow \mathbb{P}^n$.

Proof. On affine charts $U_i = \{z_i \neq 0\}$, we have trivializations:

$$\pi^{-1}(U_i) \cong U_i \times \mathbb{C}, \quad (l, z) \mapsto (l, z_i)$$

The transition functions are $\psi_{ij}(l) = \frac{z_i}{z_j}$. □

Using the group structure of $\text{Pic}(\mathbb{CP}^n)$, we define

Definition 5.15. For $k \in \mathbb{Z}$, define $\mathcal{O}(k) = \mathcal{O}(-1)^{\otimes (-k)}$, with $\mathcal{O}(-1)^{\otimes 0} := \mathcal{O}_{\mathbb{P}^n}$.

Notice that since $H^2(\mathbb{CP}^2, \mathbb{Z}) \cong \mathbb{Z}$ is torsion-free, all line bundles above are genuinely different, i.e. $\mathcal{O}(k) \cong \mathcal{O}(l)$ if and only if $k = l$, both in the holomorphic and complex vector bundle categories. Moreover, one can ask if $\mathcal{O}(-1)$ is a generator of $H^2(\mathbb{CP}^2, \mathbb{Z})$ and if not, what is its multiplicity. For now, we claim

Theorem 5.16. *The line bundle $\mathcal{O}(1)$ is a generator of $H^2(\mathbb{CP}^n, \mathbb{Z})$.*

The proof requires further work, and we will postpone it until a later section. Another class of examples of line bundles that will interest us is the following:

Definition 5.17. The *canonical bundle* of a complex manifold X is the bundle of holomorphic top-degree forms $K_X = \bigwedge^{\dim X} T^*X^{1,0}$.

There is an interesting class of compact complex manifolds, characterised by their canonical bundle:

Definition 5.18. A compact complex manifold with $K_X \cong \mathcal{O}_X$ is called (*weak*) *Calabi-Yau*.

Remark 5.19. Some authors add the further requirement that $\bigwedge^p T^*X^{1,0}$ contains no trivial subbundles for $1 \leq p < \dim X$. If you are familiar with special holonomy, this is essentially equivalent to asking the holonomy of X to be irreducible (i.e. not locally a product).

We would like to characterise the canonical bundle of the complex projective space, in virtue of Theorem 5.16. We have

Theorem 5.20 (Euler Sequence). *The holomorphic tangent bundle fits in the short exact sequence of sheaves:*

$$0 \rightarrow \mathcal{O}_{\mathbb{CP}^n} \rightarrow \bigoplus_{i=0}^n \mathcal{O}(1) \rightarrow \tau_{\mathbb{P}^n} \rightarrow 0$$

Proof. On $\mathbb{C}^{n+1} \setminus \{0\}$, take coordinates $z_0, \dots, z_n$, and $\pi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{CP}^n$ the standard projection. Let $\tilde{z}_i = \frac{z_i}{z_0}$ for $i \neq 0$ local coordinates in $\mathbb{CP}^n$. Then, we have

$$d\tilde{z}_i \left(\pi_* \frac{\partial}{\partial z_j} \right) = d \left(\frac{z_i}{z_0} \right) \left(\frac{\partial}{\partial z_j} \right) = \frac{z_0 dz_i - z_i dz_0}{z_0^2} \left(\frac{\partial}{\partial z_i} \right),$$

so

$$\pi_* \left(\frac{\partial}{\partial z_i} \right) = \frac{1}{z_0} \frac{\partial}{\partial \tilde{z}_i} \quad \pi_* \left(\frac{\partial}{\partial z_0} \right) = - \sum_{i=1}^n \frac{z_i}{z_0^2} \frac{\partial}{\partial \tilde{z}_i}.$$

Hence, for $L : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ a linear map, if we set $v_i = L \cdot \frac{\partial}{\partial z_i}$, $\pi_*(v_i)$ defines a section of $\tau_{\mathbb{CP}^n}$. In particular, $\tau_{\mathbb{CP}^n}$ is spanned $\left\{ \pi_* \left(\frac{\partial}{\partial z_i} \right) \right\}$ for $i \in \{0, \dots, n\}$, with the relation

$$\sum_{i=0}^n z_i \frac{\partial}{\partial z_i} = 0.$$

In particular, this implies the claim, where the maps in the short exact sequence are:

$$\begin{aligned} 0 \rightarrow \mathcal{O}_{\mathbb{CP}^n} &\rightarrow \bigoplus_{i=0}^n \mathcal{O}(1) \rightarrow \tau_{\mathbb{CP}^n} \rightarrow 0 \\ 1 &\mapsto (z_0, \dots, z_n) \\ (s_0, \dots, s_n) &\mapsto \pi_* \left(\sum_{i=0}^n s_i \frac{\partial}{\partial z_i} \right) \end{aligned} \quad \square$$

In particular, by taking determinants of the Euler sequence, we have

Corollary 5.21. *The canonical bundle of $\mathbb{P}^n$ is $K_{\mathbb{P}^n} = \mathcal{O}(-n-1)$.*

Proof. We have

$$\det(\tau_{\mathbb{CP}^n}) \otimes \det(\mathcal{O}_{\mathbb{CP}^n}) = \det\left(\bigoplus_{i=0}^n \mathcal{O}(1)\right) = \mathcal{O}(n+1),$$

and the claim follows from the fact that $K_X = \det(\tau_X)^*$. $\square$

Now, given a holomorphic line bundle $L \rightarrow X$, it is a natural question to study its space of sections. Let $V = \langle s_0, \dots, s_n \rangle \subseteq H^0(X, L)$ be a linear subspace of (globally defined) holomorphic sections of L . We have the following definitions.

Definition 5.22. The *base locus* of V is the vanishing locus of sections spanning $V = \langle s_0, \dots, s_n \rangle$.

$$B_L(V) := \{x \in X \mid s_0(x) = \dots = s_n(x) = 0\}.$$

The *pluricanonical map* of $V = \langle s_0, \dots, s_n \rangle$ is defined as

$$\begin{aligned} \phi_{s_0, \dots, s_n} : X \setminus B_L(V) &\rightarrow \mathbb{CP}^n \\ x &\mapsto [(\psi \circ s_0)(x) : \dots : (\psi \circ s_n)(x)], \end{aligned}$$

for ψ a local trivialisation of L and a choice of homogeneous coordinates on $\mathbb{CP}^n$.

As usual, one routinely checks that the objects above are well-defined.

Proposition 5.23. *The pluricanonical map $\phi_{s_0, \dots, s_n}$ is a well-defined holomorphic map. Moreover, for two different bases $\{s_i\}$ $\{s'_i\}$ of V , there exists a biholomorphism $\Psi : \mathbb{CP}^n \rightarrow \mathbb{CP}^n$ such that $\phi_{\{s_i\}} = \Psi \circ \phi_{\{s'_i\}}$*

This suggests that if one has a line bundle L with "enough" sections, one can hope to find a pluricanonical map such that

- it has empty base locus,
- is injective, and
- has injective differential.

Finding sufficient conditions for these conditions to be satisfied is roughly the idea behind Kodaira's embedding theorem, which we will prove in Section 10. However, we still have a long way to go before we can quantify what "enough" means.

For now, we introduce the concept of Kodaira dimension. The group structure on the space of line bundles induces a map

$$H^0(X, L_1) \otimes H^0(X, L_2) \rightarrow H^0(X, L_1 \otimes L_2).$$

Thus, we can consider the graded ring

$$R(X, L) = \bigoplus_{k \geq 0} H^0(X, L^k)$$

with the understanding that $L^0 = \mathcal{O}_X$. By the identity principle, it follows that $R(X, L)$ is an integral domain whenever X is connected. In particular, we can consider its field of fractions $Q(X, L)$. Moreover, since $R(X, L)$ is graded, we can further construct the following subfield of the field of fractions:

Definition 5.24. Let $Q^0(X, L)$ the subfield of $Q(X, L)$ that consists of elements of the form f/g with $f, g \in H^0(X, L^k)$ for some k .

The interest in $Q^0(X, L)$ is motivated by the following proposition:

Proposition 5.25. *For any line bundle $L \rightarrow X$, there is a map $Q^0(X, L) \rightarrow \mathcal{K}_X(X)$.*

Proof. Fix $k \geq 1$ and set $L' = L^k$. Consider $0 \neq s_1, s_2 \in H^0(X, L')$. We define a meromorphic function on X as follows.

Choose a trivialising cover (U_i, ψ_i) . Then $\psi_i \circ s_j$ define holomorphic functions on U , and so $f_i = \frac{\psi_i \circ s_2}{\psi_i \circ s_1}$ is a locally defined meromorphic function. To see that $\{f_i\}$ defines a global meromorphic function, it suffices to show that it is independent of the choice of trivialisation. Indeed, we have

$$f_j = \frac{\psi_j \circ s_2}{\psi_j \circ s_1} = \frac{(\psi_{ij} \circ \psi_i) \circ s_2}{(\psi_{ij} \circ \psi_i) \circ s_1} = \frac{\lambda \psi_i \circ s_2}{\lambda \psi_i \circ s_1} = f_i,$$

since $\psi_{ij} = \lambda \in \mathbb{C}^*$ since L' is a complex line bundle. □

Let us now define

Definition 5.26. Let X be a connected compact complex manifold and $L \rightarrow X$ a holomorphic line bundle. We define the *Iitaka dimension* as

$$\kappa(X, L) = \begin{cases} \text{tr. deg}_{\mathbb{C}} Q(X, L) - 1 & \text{if } Q(X, L) \neq \mathbb{C} \\ -\infty & \text{otherwise} \end{cases}.$$

If $L = K_X$ the canonical bundle, we write $\kappa(X, K_X) = \kappa(X)$ and call it the *Kodaira dimension*.

We have the following

Proposition 5.27. *For any line bundle $L \rightarrow X$, we have*

$$\kappa(X, L) \leq a(X).$$

Proof. If $\kappa(X, L) = -\infty$, there's nothing to prove. Thus, it suffices to prove $\text{tr. deg}_{\mathbb{C}} Q(R) - 1 = \text{tr. deg}_{\mathbb{C}} Q^0(R)$ for any graded ring such that $Q(R) \neq \mathbb{C}$.

First, for $f_0, \dots, f_k \in Q(R(X))$ are algebraically independent elements of degree d_i , $\frac{f_1^{e_1}}{f_0^{e_0}}, \dots, \frac{f_k^{e_k}}{f_0^{e_0}}$ with $e_i = \prod_{j \neq i} d_j$ are algebraically independent elements of $Q^0(R)$. Conversely, given $f_1, \dots, f_k \in Q^0(R)$ algebraically independent, and $f_0 \in Q(R) \setminus Q^0(R)$, then $f_0, \dots, f_k$ are algebraically independent in $Q(R)$. □

We conclude this section with the computation of the Iitaka dimensions of the line bundles $\mathcal{O}(k)$. First, we need the following result.

Proposition 5.28. *The global sections of $\mathcal{O}(k)$ are given by:*

$$H^0(\mathbb{P}^n, \mathcal{O}(k)) = \begin{cases} \mathbb{C}[z_0, \dots, z_n]_k & \text{if } k \geq 0 \\ 0 & \text{if } k < 0 \end{cases},$$

where $\mathbb{C}[z_0, \dots, z_n]_k$ denotes the space of homogeneous polynomials of degree k .

Proof. Let us prove it for $k \geq 0$. Recall that homogeneous polynomials of degree k are in one to one correspondence with k -linear symmetric forms F . Thus, a polynomial $P \in \mathbb{C}[z_0, \dots, z_n]_k$ defines a linear map $\phi_P : (\mathbb{C}^{n+1})^{\otimes k} \rightarrow \mathbb{C}$, and thus a holomorphic map $s_P : \mathbb{CP}^n \times (\mathbb{C}^{n+1})^{\otimes k} \rightarrow \mathbb{CP}^n$ that is linear on each fibre. Restricting to $\mathcal{O}(-k)$, gives a section of $\mathcal{O}(k)$.

Explicitly, for $(l; x_1, \dots, x_k) \in \mathcal{O}(-k)$, write $x_i = \lambda_i z$ for a fixed $z \in l$. Then $s_P(l; x_1, \dots, x_k) = (\prod_i \lambda_i) P(z)$. We need to show that this map is bijective. Injectivity is clear, since if $s_P \equiv 0$, the polynomial P vanishes at every line so $P = 0$.

To prove surjectivity, let $t \in H^0(\mathbb{CP}^n, \mathcal{O}(k))$ and let s_P another section induced by a polynomial of degree k . Consider the meromorphic function $F = \frac{t}{s_P} \in \mathcal{K}(\mathbb{CP}^n)$, and the associated meromorphic function on the punctured space $\tilde{F} := F \circ \pi \in \mathcal{K}(\mathbb{C}^{n+1} \setminus \{0\})$. Now, $G = P\tilde{F}$ is a homogeneous holomorphic function on $\mathbb{C}^{n+1} \setminus \{0\}$ of degree k which extends to $\mathbb{C}^{n+1}$ by Hartogs' phenomenon. By Liouville's theorem, G is a (homogeneous) polynomial of degree k , which clearly satisfies $G|_{\mathcal{O}(-k)} = t$, as needed.

The cases $k < 0$ follow by the fact that a holomorphic line bundle and its dual both admit global sections if and only if it is isomorphic to the trivial bundle (cf. Exercise Sheet). $\square$

We readily have

Corollary 5.29.

$$\kappa(\mathbb{CP}^n, \mathcal{O}(k)) = \begin{cases} n & \text{if } k > 0 \\ 0 & \text{if } k = 0 \\ -\infty & \text{if } k < 0 \end{cases}$$

Notice that the proof we gave to Proposition 4.8 corresponds precisely to the statement $\kappa(\mathbb{CP}^n, \mathcal{O}(1)) \leq a(\mathbb{CP}^n)!$

6 Divisors and blow-ups

For convenience we shall always assume that the complex manifolds we work with are connected, unless otherwise specified. Recall from Section 1, we defined analytic sets and the concept of irreducibility for analytic germs. We define

Definition 6.1. For X a complex manifold, a *divisor* on X is a formal locally finite linear combination:

$$D = \sum a_i [Y_i]$$

where Y_i are irreducible analytic hypersurfaces and $a_i \in \mathbb{Z}$. The collection of divisors with its natural group structure is called the *group of divisors* and denoted $\text{Div}(X)$.

In other words, $\text{Div}(X)$ is the free abelian group over the collection of irreducible analytic hypersurfaces. In our case, local finiteness translates to the following condition: for all $x \in X$, there exists an open neighbourhood U such that $U \cap D$ is a finite sum.

Definition 6.2. A divisor $D = \sum a_i[Y_i]$ is called *effective*, and denoted by $D \geq 0$, if all $a_i \geq 0$.

Definition 6.3. Let Y be an irreducible hypersurface, $x \in Y$, U an open neighbourhood in X and $f \in \mathcal{K}_X(U)$.

The *order* of f along Y at x , denoted by $\text{ord}_{Y,x}(f) \in \mathbb{Z}$, is defined as the unique integer such that

$$f_x = g_x^{\text{ord}_{Y,x}(f)} h_x$$

in $\mathcal{K}_{X,x}$, where $g \in \mathcal{O}_{X,x}$ is irreducible, and $h \in \mathcal{O}_{X,x}^*$. The order of f along Y , $\text{ord}_Y(f) \in \mathbb{Z}$ is the order of f at x such that Y is irreducible at x .³

Again, it follows from the good properties of the ring of germs $\mathcal{O}_{X,x}$ and Hilbert's Nullstellensatz that the order is well-defined. Moreover, it's not hard to check that it satisfies

$$\text{ord}_Y(fg) = \text{ord}_Y(f) + \text{ord}_Y(g) .$$

In particular, we get a group morphism

$$\begin{aligned} \Phi : \mathcal{K}_X^*(X) &\rightarrow \text{Div}(X) \\ f &\mapsto (f) := \sum \text{ord}_Y(f)[Y] \end{aligned} \tag{9}$$

where the sum is taken over all irreducible hypersurfaces $Y \subseteq X$.

An element in the image of Φ is called a *principal divisor*.

Proposition 6.4. *There is an isomorphism $H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \xrightarrow{\cong} \text{Div}(X)$*

Proof. Elements in $H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ are given by a collection $\{U_i, f_i\}$, where $\{U_i\}$ is a cover of X , and $f_i \in \mathcal{K}_X^*(U_i)$ satisfying $f_i f_j^{-1} \in \mathcal{O}_X^*(U_i \cap U_j)$. Let $f = \{U_i, f_i\} \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ and Y an irreducible hypersurface. We claim that $\text{ord}_Y(f)$ is well-defined.

We may assume that $Y \cap U_i \cap U_j \neq \emptyset$, otherwise there's nothing to prove. Since $f \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$, there exists $h_{ij} \in \mathcal{O}_X^*(U_i \cap U_j)$, we have

$$\text{ord}_Y(f_i) = \text{ord}_Y(h_{ij}) + \text{ord}_Y(f_j) = \text{ord}_Y(f_j) .$$

Conversely, let $D = \sum a_i[Y_i]$ be a divisor. Choose an open cover $\{U_j\}$ such that $Y_i \cap U_j = Z(g_{ij})$ for some irreducible $g_{ij} \in \mathcal{O}_X(U_j)$, which is unique up to units in $\mathcal{O}_X(U_j)$, and define

$$f_j = \prod_i g_{ij}^{a_i} .$$

Then $f_j \in \mathcal{K}_X(U_j)$ and $f_j/f_k \in \mathcal{O}_X^*(U_j \cap U_k)$ since $g_{ij}/g_{kj} \in \mathcal{O}_X^*(U_j \cap U_k)$. □

³It is implicit on the definition that the order of f at Y does not depend on the chosen point. This is indeed the case, but we shall skip the proof. See [Huy05, Prop. 1.1.35] for further details.

Using the identification above, we have:

Corollary 6.5. *There is an exact sequence:*

$$0 \rightarrow \mathbb{C}^* \rightarrow \mathcal{K}_X^*(X) \xrightarrow{\Phi} \text{Div}(X) \rightarrow \text{Pic}(X) .$$

In particular, to every divisor D , we have an associated line bundle $\mathcal{O}(D)$. The line bundle is trivial if and only if $D = (f)$ for some non-trivial meromorphic function f .

Proof. Take the long exact sequence of cohomology of the short exact sequence of sheaves:

$$0 \rightarrow \mathcal{O}_X^* \rightarrow \mathcal{K}_X^* \rightarrow \mathcal{K}_X^*/\mathcal{O}_X^* \rightarrow 0$$

□

In view of this exact sequence, we define

Definition 6.6. The *divisor class group* is

$$\text{Cl}(X) = \text{Div}(X)/\{(f) \mid f \text{ meromorphic}\}.$$

We would like to understand the image of the map $\mathcal{O} : \text{Cl}(X) \hookrightarrow \text{Pic}(X)$. While this map is, in general, not surjective, we have the following result:

Proposition 6.7. *There is the following line bundle - divisor correspondence:*

- (i) *Let $0 \neq s \in H^0(X, L)$ for a non-trivial line bundle. Then $\mathcal{O}(Z(s)) \cong L$.*
- (ii) *For any effective divisor $D \in \text{Div}(X)$, there exists $s \in H^0(X, \mathcal{O}(D))$ such that $Z(s) = D$.*

Proof.

- (i) Let $L \in \text{Pic}(X)$, and choose (U_i, φ_i) a trivialising cover. The divisor $Z(s)$ associated to $0 \neq s \in H^0(X, L)$ is given by $f := \{\varphi_i(s|_{U_i})\} \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. Then, the line bundle associated to $Z(s)$ corresponds to the cocycle $\{(U_i, f_i)\}$, but

$$f_i \cdot f_j^{-1} = \varphi_i(s|_{U_i \cap U_j}) \cdot (\varphi_j(s|_{U_i \cap U_j}))^{-1} = \varphi_i \circ \varphi_j^{-1}$$

- (ii) Let $D \in \text{Div}(X)$ be an effective divisor, represented by $\{(U_i, f_i \in \mathcal{K}_X^*(U_i))\}$. Since D is effective, the functions f_i are holomorphic, $f_i \in \mathcal{O}(U_i)$. Since the line bundle $\mathcal{O}(D)$ defined via the cocycle $\{(U_i \cap U_j, \psi_{ij} = f_i \cdot f_j^{-1})\} \in H^1(X, \mathcal{O}_X^*)$, the local holomorphic functions $f_i \in \mathcal{O}(U_i)$ define a global section $s \in H^0(X, \mathcal{O}(D))$, and $Z(s)|_{U_i} = Z(f_i) = D \cap U_i$, so $Z(s) = D$, as claimed. □

Finally, recall that, associated to a complex submanifold $Y \subseteq X$, one has the normal subbundle $\mathcal{N}_{Y|X}$, as the cokernel of the injection $\tau_Y \hookrightarrow \tau_X$. We would like to characterise the normal bundle of hypersurfaces in view of the preceding discussion. We have

Proposition 6.8. *Let $Y \subset X$ be a (smooth) hypersurface. Then:*

$$\mathcal{N}_{Y/X} \cong \mathcal{O}(Y)|_Y.$$

Proof. It suffices to prove that the normal bundle $N_{Y|\mathbb{P}^n}$ has transition functions $\frac{\partial \varphi_{ij}^n}{\partial z_n} \circ \varphi_j$, and that these are the same as those of $\mathcal{O}(d)|_Y$. $\square$

This gives a straight application in the case of $\mathbb{CP}^n$:

Theorem 6.9 (Adjunction Formula for Hypersurfaces). *If $Y \subset \mathbb{P}^n$ is a smooth hypersurface of degree d , then:*

$$K_Y \cong \mathcal{O}(d - n - 1)|_Y.$$

Proof. By taking determinants on the short exact sequence $0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_{\mathbb{CP}^n} \rightarrow \mathcal{N}_{Y|\mathbb{CP}^n} \rightarrow 0$, we have $K_Y = K_{\mathbb{P}^n}|_Y \otimes \det(N_{Y|\mathbb{P}^n})$. But $N_{Y|\mathbb{P}^n} = \mathcal{O}(d)|_Y$, by assumption and $K_{\mathbb{P}^n} \cong \mathcal{O}(-n - 1)$ by Corollary 5.21. Hence:

$$K_Y = \mathcal{O}(-n - 1)|_Y \otimes \mathcal{O}(d)|_Y = \mathcal{O}(d - n - 1)|_Y. \quad \square$$

6.1 Blow-ups

We conclude this section by introducing a key construction in complex geometry: blow-ups. These give rise to the active field within complex and algebraic geometry, known as birational geometry. We will only discuss the fundamental construction and a few direct consequences.

For the entire section, X will be a connected complex manifold and $Y \subset X$ a closed analytic set. The blow-up of X along Y is a triple $(\text{Bl}_Y(X), E, \sigma)$, with $\hat{X} = \text{Bl}_Y(X)$ a complex manifold, $E \subseteq \hat{X}$ a divisor called the exceptional divisor, and a proper holomorphic map $\sigma : \hat{X} \rightarrow X$ such that

- (i) The map σ restricted to $\hat{X} \setminus \sigma^{-1}(Y)$ is a biholomorphism to $X \setminus Y$, and
- (ii) The map $\sigma : \sigma^{-1}(Y) \rightarrow Y$ is biholomorphic to $\mathbb{P}(\mathcal{N}_{Y|X}) \rightarrow Y$.

The blow-up map has a characterising universal property, which we will not prove.

Theorem 6.10 (Universal property of a blow-up). *Let $f : Z \rightarrow X$ a bimeromorphic map such that f restricted to $Z \setminus f^{-1}(Y)$ is holomorphic. Then there exists a unique $g : Z \rightarrow \hat{X}$ such that the diagram commutes:*

If one believes the universal characterisation of a blow-up, it is clear that the blow-up is unique, up to a unique biholomorphism. Thus, it suffices to show existence of a blow-up, to which we will devote the rest of this section.

If one disregards the universal property, the construction of a blow-up outlined below can be taken to be the definition of *the* blow-up of X along Y . We begin by considering the example of a point. Recall that the total space of the line bundle $\pi : \mathcal{O}(-1) \rightarrow \mathbb{CP}^n$ is the incidence variety inside $\mathbb{C}^{n+1} \times \mathbb{CP}^n$. Let us consider the other projection $\sigma : \mathcal{O}(-1) \rightarrow \mathbb{C}^{n+1}$. For $z \neq 0$ the pre-image $\sigma^{-1}(z)$ is the unique line l_z passing through $z \in \mathbb{C}^{n+1}$. However, the preimage at zero is the entire complex projective space, $\sigma^{-1}(0) = \mathbb{CP}^n$, as any line in $\mathbb{C}^{n+1}$ goes through the origin $0 \in \mathbb{C}^{n+1}$. In fact, $\sigma^{-1}(0)$ is simply the zero section of the line bundle $\pi : \mathcal{O}(-1) \rightarrow \mathbb{CP}^n$.

We define the blow-up of 0 in $\mathbb{C}^{n+1}$ as the total space of the line bundle $(\mathcal{O}(-1), \pi^{-1}(0), \sigma)$, the total space of $\mathcal{O}(-1)$, where the zero section $\pi^{-1}(0)$ is the exceptional divisor E , together with the

natural projection $\sigma : \mathcal{O}(-1) \rightarrow \mathbb{C}^{n+1}$. Note that, σ is a biholomorphism away from the origin, whilst the normal bundle of 0 is simply $\mathbb{C}^{n+1}$, so $\sigma|_{\pi^{-1}(0)} : \mathbb{P}(\mathbb{C}^{n+1}) = \mathbb{CP}^n \rightarrow \{0\}$, as needed. For an arbitrary linear subspace $\mathbb{C}^m \subseteq \mathbb{C}^{n+1}$, consider

$$Bl_m(\mathbb{C}^{n+1}) := \{(z, l) \in \mathbb{C}^{n+1} \times \mathbb{CP}^{n-m} \mid z \in \langle \mathbb{C}^m, l \rangle\}.$$

Clearly, $Bl_m(\mathbb{C}^{n+1}) \rightarrow \mathbb{CP}^{n-m}$ is a $\mathbb{C}^{m+1}$ -fibre bundle, and using the same argument as in the proof of Proposition 5.14, it is a holomorphic bundle, so the total space $Bl_m(\mathbb{C}^{n+1})$ is a complex manifold. The projection $\sigma : Bl_m(\mathbb{C}^{n+1}) \rightarrow \mathbb{C}^{n+1}$ gives the required blow up.

Let us construct the blow-up of a complex manifold along a submanifold $Y^m \subset X^n$. Of course, the idea is to use the previous construction as a local model and glue along different coordinate charts.

Proposition 6.11. *Let Y be a complex submanifold of X . Then the blow-up of X along Y exists.*

Proof. Take $\{U_i, \varphi_i\}$ an atlas of X such that $\varphi_i(U_i \cap Y) = \varphi(U_i) \cap \mathbb{C}^m \subseteq \mathbb{C}^n$, and consider $\sigma : Bl_{\mathbb{C}^m}(\mathbb{C}^n) \rightarrow \mathbb{C}^n$ the blow-up of $\mathbb{C}^n$ along $\mathbb{C}^m$ as constructed above (note that we had $n+1$ above, rather than n).

We denote by $\sigma_i : Z_i \rightarrow \varphi_i(U_i)$ the restriction of the blow-up to the open subset $\varphi_i(U_i) \subset \mathbb{C}^n$, i.e. $Z_i = \sigma^{-1}(\varphi_i(U_i))$, and $\sigma_i = \sigma|_{Z_i}$. The goal is to prove that the "local" blow-ups glue along different charts.

[TO BE ADDED]

□

Proposition 6.12. *The canonical bundle $K_{\hat{X}}$ of the blow-up $(\hat{X}, E, \sigma)$ is isomorphic to $\sigma^*K_X \otimes \mathcal{O}_{\hat{X}}((n-1)E)$.*

Proof. [TO BE ADDED]

□

Corollary 6.13. *For $E = \mathbb{CP}^{n-1} \subset \hat{X} \rightarrow X$, one has $\mathcal{O}(E)|_E \cong \mathcal{O}(-1)$.*

Proof. By the previous proposition $K_{\hat{X}} \cong \sigma^*K_X \otimes \mathcal{O}((n-1)E)$, and by the adjunction formula $K_{\mathbb{CP}^{n-1}} \cong (K_{\hat{X}} \otimes \mathcal{O}(E))|_E$. Hence, $K_{\mathbb{CP}^{n-1}} \cong \mathcal{O}(nE)$. Since $K_{\mathbb{CP}^{n-1}} \cong \mathcal{O}(=n)$ by Corollary 5.21 and $\text{Pic}(\mathbb{CP}^{n-1}) \cong \mathbb{Z}$ is torsion free, the claim follows. □

7 Hermitian metrics and connections

The idea now is to combine our previous discussion with the choice of metric on our bundles. First, we go through some basic linear algebra results.

Definition 7.1. A *hermitian inner product* on a complex vector space E is a bilinear map $\langle \cdot, \cdot \rangle : E \otimes \bar{E} \rightarrow \mathbb{C}$ such that,

- $h(a, \bar{b}) = \overline{h(b, \bar{a})}$ (Hermitian symmetry)
- $h(a, \bar{a}) \geq 0$ with equality iff $a = 0$ (positive definiteness)

In particular, a Hermitian metric induces an anti-linear isomorphism $E \cong \overline{E}^*$. The following lemmas are standard linear algebra:

Lemma 7.2. *Let E be a complex vector space. The following objects are in one-to-one correspondence:*

- (i) *Hermitian inner products h ,*
- (ii) *(Real) inner products compatible with the complex structure J : $g(\cdot, \cdot) = g(J\cdot, J\cdot)$*
- (iii) *Non-degenerate positive real $(1, 1)$ -forms, ω*

Lemma 7.3. *Let V be an n -dimensional complex vector space, and g a compatible inner product. Consider $\omega = g(J \cdot, \cdot)$ the associated $(1, 1)$ form, then*

$$\mathrm{dvol}_g = \frac{\omega^n}{n} .$$

In particular, if $W \subseteq V$ is a m -dimensional complex subspace, we have

$$\mathrm{vol}_g(Y) = \frac{\omega^m}{m!}$$

In fact, this is a characterising property, due to Wirtinger:

Lemma 7.4 (Wirtinger inequality). *Let $W^{2k} \subseteq (V^{2n}, g, J)$ be a subspace of a Euclidean complex vector space. Denote by ω the associated $(1, 1)$ -form. Then*

$$\left. \frac{\omega^k}{k!} \right|_W \leq \left. \mathrm{vol}_g \right|_W ,$$

with equality if and only if W is a complex subspace.

Proof. Let $\{e_{2i-1}, e_{2i}\}$ be an orthonormal basis of W and $\{v_{2i-1}, v_{2i}\}$ its dual basis. Denote by $\iota : W \hookrightarrow V$ the inclusion map. Then,

$$\iota^* \omega = \sum_{i=1}^k \omega(e_{2i-1}, e_{2i}) v_{2i-1} \wedge v_{2i} .$$

Thus,

$$\iota^* \left(\frac{\omega^k}{k!} \right) = \prod_{i=1}^k \omega(e_{2i-1}, e_{2i}) \mathrm{vol}_g = \prod_{i=1}^k g(Je_{2i-1}, e_{2i}) \mathrm{vol}_g \leq \mathrm{vol}_g ,$$

where the last inequality is simply the Cauchy-Schwarz inequality. The equality case implies $Je_{2i-1} = \pm e_{2i}$, as needed. $\square$

On a Euclidean vector space, any form $\varphi \in \bigwedge^k V$ satisfying that $\varphi|_W \leq \mathrm{vol}_W$ for all k -planes $W \subseteq V$ is called a *(pre)calibration*.

Let us study the linear algebra associated to a hermitian vector space and its associated exterior algebra. Let (V^{2n}, g, J) be a hermitian vector space. Recall that the space of linear maps that

preserve the hermitian structure of V is a compact Lie group of dimension n^2 , called the unitary group $U(V)$. The space V is naturally an irreducible $U(V)$ -representation, called the standard representation. The complexified $V_{\mathbb{C}} = V \otimes \mathbb{C}$ splits as two complex irreducible representations: $V^{1,0} \oplus V^{0,1}$ as discussed above.

We are interested in understanding how $\bigwedge^k V^*$ splits into irreducible $U(V)$ -representations. First, we need the following concepts:

Definition 7.5. Let (V^{2n}, g) be a Euclidean vector space. We define the *Hodge star* operator $*$: $\bigwedge^k V^* \rightarrow \bigwedge^{2n-k} V^*$ by the universal property

$$\alpha \wedge * \beta = g(\alpha, \beta) \text{vol}_g .$$

It is elementary to check that $*$ is an isometry in $\bigwedge^\bullet V^*$ satisfying $*^2 = (-1)^{k(2n-k)} = (-1)^k$ on $\bigwedge^k V^*$.

The Hodge star extends $\mathbb{C}$ -linearly to $\bigwedge^\bullet V_{\mathbb{C}}^*$. With respect to the complex (p, q) -decomposition, we then have $*$: $\bigwedge^{p,q} \rightarrow \bigwedge^{n-q, n-p}$.

Definition 7.6. Let (V^{2n}, g, J) be a Hermitian vector space, with fundamental form ω . We define the *Lefschetz operator* by:

$$L : \bigwedge^{(p,q)} V_{\mathbb{C}}^* \bigwedge^{(p+1,q+1)} V_{\mathbb{C}}^* \\ \alpha \mapsto \alpha \wedge \omega .$$

and its adjoint $\Lambda : \bigwedge^{(p,q)} V_{\mathbb{C}}^* \rightarrow \bigwedge^{(p-1,q-1)} V_{\mathbb{C}}^*$.

Lemma 7.7. We have $\Lambda = *L*$.

Proof. By definition

$$g(\Lambda \alpha, \beta) \text{vol} = g(\alpha, L\beta) \text{vol} = L(\beta) \wedge * \alpha = \beta \wedge \omega \wedge * \alpha = (-1)^k g(\beta, * [L(*\alpha)]) \text{vol} .$$

□

We will also need the *counting operator* $H|_{\bigwedge^k V_{\mathbb{C}}^*} = (k - n) \text{Id}$. With this in hand, we have

Proposition 7.8. Let (V^{2n}, g, J) be a Hermitian vector space, with fundamental form ω . The Lefschetz operator satisfies

$$[H, L] = 2L \quad [H, \Lambda] = -2\Lambda \quad [L, \Lambda] = H .$$

In particular, $\langle L, \Lambda, H \rangle$ induce an $\mathfrak{sl}(2, \mathbb{C})$ -representation on $\bigwedge^\bullet V_{\mathbb{C}}^*$.

Proof. The statements $[H, L] = 2L$ and $[H, \Lambda] = -2\Lambda$ are immediate. Let us prove $[L, \Lambda] = H$ by induction over the dimension of V .

If we decompose $V = W_1 \oplus W_2$ complex subspaces, we have $\bigwedge^\bullet V^* = \bigwedge^\bullet W_1^* \otimes \bigwedge^\bullet W_2^*$ and $\omega = \omega_1 \oplus \omega_2$, so $L = \omega_1 \otimes 1 + 1 \otimes \omega_2 =: L_1 + L_2$. By linearity, it suffices to check the claim on split forms. Let $\alpha = \alpha_1 \otimes \alpha_2$ and $\beta = \beta_1 \otimes \beta_2$. Then,

$$\begin{aligned} g(\alpha, L\beta) &= g(\alpha, L_1\beta_1 \otimes \beta_2) + g(\alpha, \beta_1 \otimes L_2\beta_2) \\ &= g(\alpha_1, L_1\beta_1)g(\alpha_2, \beta_2) + g(\alpha_1, \beta_1)g(\alpha_2, L_2\beta_2) \\ &= g(\Lambda_1\alpha_1, \beta_1)g(\alpha_2, \beta_2) + g(\alpha_1, \beta_1)g(\Lambda_2\alpha_2, \beta_2) = g(\Lambda\alpha, \beta). \end{aligned}$$

So $\Lambda = \Lambda_1 + \Lambda_2$. Thus, by the induction hypothesis,

$$[L, \Lambda](\alpha) = H_1(\alpha_1) \otimes \alpha_2 + \alpha_1 \otimes H_2(\alpha_2) = (k_1 - n_1)\alpha_1 \otimes \alpha_2 + (k_2 - n_2)\alpha_1 \otimes \alpha_2 = (k - n)\alpha.$$

Thus, the base case $n = 1$ remains. Let $\{x, y\}$ be a basis of $V \cong \mathbb{C}$ with $Jx = y$, so

$$\bigwedge^\bullet V^* = \bigwedge_{\mathbb{R}}^0 V^* \oplus \bigwedge_{\langle x, y \rangle}^1 V^* \oplus \bigwedge_{\langle \omega \rangle}^2 V^*$$

Notice that L and Λ act trivially on $\bigwedge^1 V^*$ by degree reasons. Finally, one checks that

$$[L, \Lambda](\lambda) = -\Lambda(\lambda\omega) = -\lambda, \quad [L, \Lambda](\mu\omega) = L\Lambda(\mu\omega) = \mu\omega$$

for $\lambda, \mu \in \mathbb{R}$. □

Using induction, one gets

Corollary 7.9. *For $i \geq 1$, we have $[L^i, \Lambda](\alpha) = i(k - n + i - 1)L^{i-1}(\alpha)$.*

Definition 7.10. Let (V, g, J) be a Hermitian vector space and consider the associated operators L, Λ and H . A k -form $\alpha \in \bigwedge^k V^*$ is called *primitive* if $\Lambda(\alpha) = 0$. The subspace of primitive k -forms is denoted P^k .

Proposition 7.11 (Lefschetz decomposition). *Let (V, g, J) be a Hermitian vector space and consider the associated operators L, Λ and H . We have a direct sum decomposition*

$$\bigwedge^k V^* = \bigoplus_{i \geq 0} L^i(P^{k-2i}).$$

Finally, note that the complex structure J extends naturally to the space of k -forms: for $\alpha \in \bigwedge^k V$, we define $J(\alpha)(v_1, \dots, v_k) := \alpha(Jv_1, \dots, Jv_k)$. The relation between I, L and the Hodge star is made precise by the following result, due to Weil:

Lemma 7.12. *Let (V, g, J) be a Hermitian vector space. Consider the operators:*

- $\star = (-1)^{\binom{k}{2} + k} J$
- $\Theta = \exp(L) \exp(-\Lambda) \exp(L)$.

Then $\star = \Theta$.

Proof. We claim that the same dimensional induction argument used in the proof of Proposition 7.8 works in this setup. We leave it to the reader to verify the details.

Thus, it suffices to prove this for a complex one-dimensional space. As above, let $\{x, y\}$ be a basis of $V \cong \mathbb{C}$ with $Jx = y$, so

$$\bigwedge^{\bullet} V^* = \bigwedge^0 V^* \oplus \bigwedge^1 V^* \oplus \bigwedge^2 V^* \\ \mathbb{R} \qquad \langle x, y \rangle \qquad \langle \omega \rangle$$

For degree 0, we have $\star 1 = (-1)^{\binom{0}{2}} * I(1) = \omega$, and

$$\begin{aligned} \Theta(1) &= \exp(L) \exp(-\Lambda) \exp(L)(1) = \exp(L) \exp(-\Lambda)(1 + \omega) \\ &= \exp(L)(1 + \omega - \Lambda\omega) = \exp(L)(\omega) = \omega . \end{aligned}$$

For degree 1, we have $\star x = - * I(x) = - * y = x$ and $\star y = - * I(y) = * x = y$, and by (bi)degree reasons, $\Theta = \text{Id}$. Finally, for degree 2, we have $\star \omega = -\omega$ and

$$\begin{aligned} \Theta(\omega) &= \exp(L) \exp(-\Lambda) \exp(L)(\omega) = \exp(L) \exp(-\Lambda)(\omega) \\ &= \exp(L)(\omega - 1) = -\omega . \end{aligned} \quad \square$$

As a corollary of the equality $\star = \Theta$, we get the following useful identity, known as Weil's formula:

Corollary 7.13 (Weil's formula). *For all $\alpha \in P^k$, we have*

$$*L^j(\alpha) = (-1)^{\binom{k}{2}} \frac{j!}{(n-k-j)!} L^{n-k-j}(J(\alpha)) ,$$

Proof. Note that since $\Lambda = - * L*$, we have $* \exp(L) = \exp(\Lambda)$ and $* \exp(\Lambda) = \exp(L)*$. Thus, for $\alpha \in P^k$, we have

$$\begin{aligned} \exp(L) \exp(-\Lambda) \exp(L)\alpha &= (-1)^{\binom{k}{2}+k} * I(\alpha) \\ \exp(\Lambda) \exp(-L) * \exp(L)\alpha &= (-1)^{\binom{k}{2}} I(\alpha) \\ * \exp(L)\alpha &= (-1)^{\binom{k}{2}} \exp(L) \exp(-\Lambda) I(\alpha) \\ &= \end{aligned}$$

□

Finally, we have

Definition 7.14. Let (V, g, J) be a Hermitian vector space. For $k \leq n$, the *Hodge—Riemann* pairing is defined as

$$\begin{aligned} \bigwedge^k V_{\mathbb{C}}^* \times \bigwedge^k V_{\mathbb{C}}^* &\rightarrow \mathbb{C} \\ (\alpha, \beta) &\mapsto (-1)^{\frac{k(k+1)}{2}} \alpha \wedge \beta \wedge \omega^{n-k} \end{aligned}$$

Proposition 7.15. *Let $\alpha \in \Lambda^{p,q} V_{\mathbb{C}}^*$ and $\beta \in \Lambda^{p',q'} V_{\mathbb{C}}^*$.*

(i) The Hodge-Riemann pairing vanishes unless $(p, q) = (q', p')$.

(ii) For $0 \neq \alpha \in P^{p,q} \subseteq \Lambda^{p,q} V_{\mathbb{C}}^*$, we have

$$i^{p-q} Q(\alpha, \bar{\alpha}) = [n - (p + q)]! \langle \alpha, \alpha \rangle > 0.$$

We leave the proof as an exercise to the reader.

Of course, we are interested vector bundles rather than vector spaces:

Definition 7.16. A *hermitian metric* on a complex vector bundle E is a smooth section of $(E \otimes \bar{E})^*$ such that it induced a hermitian structure on each fibre.

By using a partition of unity subordinate to a trivialisation of E , every complex vector bundle admits a Hermitian metric, as in the Riemannian case. As usual, we have

Lemma 7.17. If (E, h) and (F, h') are Hermitian vector bundles, then $E \otimes F$, $\text{Hom}(E, F)$, $\bigwedge^p E$ inherit natural Hermitian metrics.

We denote the standard hermitian structure on $\mathbb{C}$ by $\langle \cdot, \cdot \rangle$.

Example 7.18. 1. For $E = \bigoplus_{i=1}^k \mathcal{O}_X$, we have $h(s, t)(x) = \sum_{i=1}^k \langle s_i(x), t_i(x) \rangle$.

2. For a line bundle $L \rightarrow X$ with empty base locus and a basis of global sections $s_1, \dots, s_k$, we can define:

$$h(\zeta, \xi)(x) = \frac{\langle \psi(\zeta), \psi(\xi) \rangle}{\sum |\psi(s_i)|^2}$$

where ψ is a local trivialization.

We are working with complex vector bundles (which are additionally holomorphic), but not all properties given by the choice of a hermitian metric extend to the complex category.

For instance, while it is true that given a short exact sequence of holomorphic vector bundles and a hermitian metric on the middle term, the sequence naturally splits in the complex bundle category, it does not split in the category of holomorphic bundles.

For instance, if one takes the $\mathcal{O}(-2)$ -twisted Euler sequence in $\mathbb{CP}^1$:

$$0 \rightarrow \mathcal{O}(-2) \rightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathcal{O} \rightarrow 0,$$

then $H^0(\mathbb{CP}^1, \mathcal{O}(-1)^{\oplus 2}) = 0 \neq \mathbb{C} = H^0(\mathcal{O} \oplus \mathcal{O}(-2))$.

Finally, all the computations and results on the Lefschetz operators and related discussion carry over naturally to the case of complex vector bundles.

7.1 Hodge Theory

The choice of a compatible Riemannian metric on a complex manifold X induces hermitian metrics on the bundles $\bigwedge^k T^*X$ and $\bigwedge^{p,q} T^*X$, and so the spaces $\Omega^k(M)$ and $\Omega^{p,q}(M)$ are equipped with the usual inner product and L^2 -norm.

As in the smooth case, one can ask:

Question 7.19. Given a class $[\psi] \in H_{\bar{\partial}}^{p,q}(X)$, is there a representative with minimal L^2 -norm?

As expected, the answer to this question is given by the L^2 -adjoint of the corresponding operator:

Lemma 7.20. Let $\bar{\partial}^*$ be the L^2 -adjoint to $\bar{\partial}$. Then ψ with $\bar{\partial}\psi = 0$ has minimal L^2 -norm if and only if $\bar{\partial}^*\psi = 0$.

Proof. First, assume we have $\psi \in \mathcal{A}_X^{p,q}$ with $\bar{\partial}\psi = 0 = \bar{\partial}^*\psi$. Then, for any other representative $\tilde{\psi} = \psi + \bar{\partial}_E\eta$, we have

$$\|\tilde{\psi}\|^2 = \|\psi\|^2 + \|\bar{\partial}\eta\|^2 + 2\operatorname{Re}\langle\psi, \bar{\partial}\eta\rangle = \|\psi\|^2 + \|\bar{\partial}\eta\|^2 + 2\operatorname{Re}\langle\bar{\partial}^*\psi, \eta\rangle \geq \|\psi\|^2.$$

Conversely, assume ψ has minimal norm. In particular, for all $\eta \in \mathcal{A}_X^{p,q-1}$, we have $\frac{d}{dt}\|\psi + t\bar{\partial}\eta\|^2 = 0$. Differentiating, we have $\operatorname{Re}\langle\bar{\partial}^*\psi, \eta\rangle = 0$. By taking $\eta' = i\eta$, it follows that $\langle\bar{\partial}^*\psi, \eta\rangle = 0$. Since this holds for arbitrary η , we have $\bar{\partial}^*\psi = 0$. $\square$

The reader might have noticed that, while the statement is technically true, it requires more care than what has been put into the proof. Indeed, the L^2 -adjoint is only defined on the L^2 -completion of $\mathcal{A}_X^{p,q}$. However, from Stokes' theorem, one has

Lemma 7.21. For $\psi \in \mathcal{A}_X^{p,q}$, we have $\bar{\partial}^*\psi = -\bar{*}\bar{\partial}\bar{*}\psi \in \mathcal{A}_X^{p,q-1}$.

Proof. Let $\alpha \in \mathcal{A}^{p,q}$, $\beta \in \mathcal{A}^{p,q-1}$. Then, we have

$$\langle\alpha, \bar{\partial}\beta\rangle = \int_X \bar{\partial}\beta \wedge \bar{*}\alpha = \int_X \bar{\partial}(\beta \wedge \bar{*}\alpha) + (-1)^{p+q+1} \int_X \beta \wedge \bar{\partial}\bar{*}\alpha = -\langle\beta, \bar{*}\bar{\partial}\bar{*}\alpha\rangle$$

where we used that $\bar{\partial}(\beta \wedge \bar{*}\alpha) = d(\beta \wedge \bar{*}\alpha)$ since $\beta \wedge \bar{*}\alpha \in \mathcal{A}_X^{n,n-1}$. $\square$

Thus, we have a map $\ker \bar{\partial} \cap \ker \bar{\partial} \rightarrow H^{p,q}(X)$. We define the $\bar{\partial}$ -Laplacian operator

$$\begin{aligned} \Delta_{\bar{\partial}} : \mathcal{A}^{p,q} &\rightarrow \mathcal{A}^{p,q} \\ \gamma &\mapsto (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})\gamma \end{aligned}$$

and the space of $\bar{\partial}$ -harmonic forms, $\mathcal{H}_{\bar{\partial}}^{p,q} := \ker \Delta_{\bar{\partial}}$; and similarly for ∂ .

We have the following key result, due to Hodge:

Theorem 7.22 (Hodge decomposition). Let X be a compact hermitian manifold. Then there exists a natural orthogonal decomposition

$$\mathcal{A}_X^{p,q} = \bar{\partial}\mathcal{A}_X^{p,q-1} \oplus \mathcal{H}_{\bar{\partial}}^{p,q}(X) \oplus \bar{\partial}^*\mathcal{A}_X^{p,q+1}.$$

The spaces of $\bar{\partial}$ -harmonic (p,q) -forms $\mathcal{H}^{p,q}$ are finite-dimensional.

Corollary 7.23. The map $\mathcal{H}_{\bar{\partial}}^{p,q}(X) \rightarrow H_{\bar{\partial}}^{p,q}$ is an isomorphism.

Proof. It suffices to prove that $\ker \bar{\partial} = \bar{\partial}(\mathcal{A}_X^{p,q-1}) \oplus \mathcal{H}_{\bar{\partial}}^{p,q}(X)$.

Indeed, by the Hodge decomposition, if $\bar{\partial}\bar{\partial}^*\beta = 0$, we have $0 = \langle\bar{\partial}\bar{\partial}^*\beta, \beta\rangle = \|\bar{\partial}^*\beta\|^2$, as needed. $\square$

Analogously to the Poincaré duality, we have

Proposition 7.24 (Serre duality). *For X compact, the pairing*

$$\begin{aligned} \mathcal{A}^{p,q}(X) \times \mathcal{A}^{n-p,n-q}(X) &\rightarrow \mathbb{C} \\ (\alpha, \beta) &\mapsto \int_X \alpha \wedge \beta \end{aligned}$$

induces a non-degenerate pairing

$$\mathcal{H}^{p,q}(X) \times \mathcal{H}^{n-p,n-q}(X) \rightarrow \mathbb{C}$$

7.2 Connections

We want to extend the previous discussion to the more general case of E -valued forms for a vector bundle E . If E is a holomorphic bundle, we saw that there exists a natural operator $\bar{\partial}_E$ that extends the natural $\bar{\partial}$ operator. But there is no (a priori) natural extension candidate for ∂ .

To consider this extension as well as treat the more general case of complex vector bundles, we need to introduce the concept of a connection:

Definition 7.25. Consider a vector bundle $E \rightarrow X$ and $\mathcal{E}$ its associated sheaf of sections. A *connection* on E is a $\mathbb{C}$ -linear map of sheaves:

$$\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{A}_X^1,$$

satisfying the Leibniz rule:

$$\nabla(fs) = df \otimes s + f\nabla s$$

for $s \in \mathcal{E}(U)$, $f \in \mathcal{C}^\infty(U)$.

Using the natural splitting $\mathcal{A}_X^1 \cong \mathcal{A}_X^{1,0} \oplus \mathcal{A}_X^{0,1}$, we get a splitting $\nabla = \nabla^{1,0} \oplus \nabla^{0,1}$ for any connection.

Definition 7.26. Let (E, h) be a holomorphic vector bundle equipped with a hermitian metric h . A connection ∇ is called

- (i) compatible if $\nabla^{0,1} = \bar{\partial}_E$,
- (ii) metric if $\nabla h = 0$. That is, for any sections $s_1, s_2 \in \mathcal{E}$, we have

$$d(h(s_1, s_2)) = h(\nabla s_1, s_2) + h(s_1, \nabla s_2).$$

These definitions should be reminiscent of the fundamental theorem of Riemannian geometry, where the Levi-Civita connection is characterised by being the unique torsion-free metric connection. Indeed, one has

Proposition 7.27. *Let (E, h) be a holomorphic vector bundle equipped with a hermitian metric h . There exists a unique compatible metric connection on (E, h) , called the Chern connection.*

Proof. Let $h(s_i, s_j) = h_{ij}$ with s_i a local frame. If ∇ exists, ω must be of type $(1, 0)$ and

$$\begin{aligned}\partial h_{ij} + \bar{\partial} h_{ij} &= d(h_{ij}) = h(\nabla s_i, s_j) + h(s_i, \nabla s_j) \\ &= h\left(\sum \omega_{ik} \otimes s_k, s_j\right) + h\left(s_i, \sum \omega_{jl} \otimes s_l\right) \\ &= \sum_k \omega_{ik} h_{kj} + \sum_l \bar{\omega}_{jl} h_{il}\end{aligned}$$

In coordinate-free notation, we have $\partial h = \omega h$ and $\bar{\partial} h = h \bar{\omega}^T$, and there is a unique solution: $\omega = \partial h h^{-1}$. $\square$

Given a vector bundle E with a connection $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{A}_X^1$, we can extend it naturally to

$$\begin{aligned}\nabla : \mathcal{A}_X^k \otimes \mathcal{E} &\rightarrow \mathcal{A}_X^{k+1} \otimes \mathcal{E} \\ \omega \otimes s &\mapsto d\omega \otimes s + (-1)^k \omega \wedge \nabla s.\end{aligned}$$

Definition 7.28. The *curvature* of a connection ∇ is the operator:

$$\nabla^2 = \nabla \circ \nabla : \mathcal{E} \rightarrow \mathcal{A}_X^2 \otimes \mathcal{E}$$

Our interest in the curvature operator is motivated by the following proposition:

Proposition 7.29. *The curvature operator is function-linear. In particular, one can associate with it an element $F_\nabla \in \mathcal{A}^2(\text{End}(E))$ such that*

$$\nabla^2 s = F_\nabla \cdot s.$$

The $\cdot$ represents the natural action of $\text{End}(E)$ and will be omitted in the future.

Proof. Let $f \in \mathcal{A}_X^0(U)$ and $s \in \mathcal{E}(U)$ for some open U . Then

$$\nabla^2(fs) = \nabla(df \otimes s + f\nabla s) = (d^2f \otimes s - df \otimes \nabla s) + (df \otimes \nabla s + f\nabla^2 s) = f\nabla^2 s. \quad \square$$

Proposition 7.30. *Let (E, h) be a holomorphic hermitian vector bundle and ∇ its Chern connection. Then $F_\nabla \in \mathcal{A}_X^{1,1}(\text{End}(E))$.*

We conclude by showing:

Lemma 7.31. *Let (E, h) be a holomorphic vector bundle equipped with a hermitian metric, and ∇ a connection on E . Then,*

(i) *If ∇ is a compatible connection, $F_\nabla \in \mathcal{A}_X^{2,0} \oplus \mathcal{A}_X^{1,1}$.*

(ii) *If ∇ is metric, for any sections s_1, s_2 , we have*

$$h(F_\nabla(s_1), s_2) + h(s_1, F_\nabla(s_2)) = 0.$$

(iii) *The Chern connection satisfies $F_\nabla \in \mathcal{A}_X^{1,1}$.*

Proof.

(i) For any connection, the splitting $\nabla = \nabla^{1,0} + \nabla^{0,1}$ yields

$$\nabla^2 = (\nabla^{1,0})^2 + (\nabla^{1,0}\nabla^{0,1} + \nabla^{0,1}\nabla^{1,0}) + (\nabla^{0,1})^2 ,$$

where $(\nabla^{0,1})^2$ is the only $(0,2)$ component. If ∇ is compatible $\nabla^{0,1} = \bar{\partial}_E$, and the claim follows since $\bar{\partial}_E^2 = 0$ (cf. Proposition 5.6).

(ii) If ∇ is metric, we have

$$\begin{aligned} 0 &= d^2 h(s_1, s_2) = d(h(\nabla s_1, s_2) + h(s_1, \nabla s_2)) \\ &= [h(F_\nabla s_1, s_2) - h(\nabla s_1, \nabla s_2)] + [h(\nabla s_1, \nabla s_2) + h(s_1, F_\nabla s_2)] \\ &= h(F_\nabla s_1, s_2) + h(s_1, F_\nabla s_2) . \end{aligned}$$

(iii) Combining the two previous statements, the claim follows. $\square$

Finally, let us say a few more words about connections for completeness. A key property of connections is the so-called Bianchi identity:

Proposition 7.32 (Bianchi identity). *Let ∇ a connection on E . Then the curvature F_∇ satisfies $\nabla^{\text{End}} F_\nabla = 0$, where ∇^{End} is the induced connection on the endomorphism bundle.*

Since there is little risk of confusion, in the future we will be abusing notation and using ∇ to denote the connection on E as well as the induced connection on its endomorphism bundle $\text{End}(E)$.

Proof. Given the connection ∇ on E , the induced connection ∇^{End} is given by $\nabla^{\text{End}}(f)(s) = \nabla(f(s)) - f(\nabla(s))$. Using that $F_\nabla = \nabla^2$, we have

$$(\nabla^{\text{End}} F_\nabla)(s) = \nabla(F_\nabla(s)) - F_\nabla(\nabla(s)) = \nabla(\nabla^2(s)) - \nabla^2(\nabla(s)) = 0 . \quad \square$$

7.3 The first Chern class

We conclude this section by revisiting the first Chern class of a line bundle and giving an alternative interpretation of it.

For a line bundle L , we have $\text{End}(L) = L \otimes L^* = \mathbb{C}$, so the curvature F_∇ can be identified with a section of $\mathcal{A}_X^2$. Moreover, by the Bianchi identity, Proposition 7.32, we see that $dF_\nabla = 0$, so we can consider its associated cohomology class $[F_\nabla]$. We have the following theorem:

Theorem 7.33. *Let $L \rightarrow X$ be a line bundle and ∇ a connection on it. Then*

$$[F_\nabla] = 2\pi i \, c_1(L)_\mathbb{R}$$

where $c_1(L) \in H^2(X, \mathbb{Z})$ is the first Chern class of L defined by the connecting map in the exponential long exact sequence, Equation (8), and $c_1(L)_\mathbb{R} = c_1(L) \otimes \mathbb{R}$. In particular, $[F_\nabla]$ is independent of the chosen connection.

Proof. We want to construct a Čech cocycle describing $c_1(L)$. Let $\{(U_i, \phi_i)\}$ a trivialising cover of L with $U_{ij} = U_i \cap U_j$ simply connected. Then $[L] = [\{\phi_{ij}\}] \in H^1(X, \mathcal{A}_X^{1*})$.

Thus, if we set $\psi_{ij} = \log(\phi_{ij})$, we have

$$c_{ijk} = \frac{1}{2\pi i} (\psi_{ij} + \psi_{jk} - \psi_{ik})$$

defines a cocycle representing $c_1(L)$.

Recall the de Rham resolution $\mathbb{C} \rightarrow \mathcal{A}_X^0 \rightarrow \mathcal{A}_X^1 \rightarrow \dots$ is acyclic. Consider the induced short exact sequences:

$$\begin{aligned} 0 \rightarrow \mathbb{Z} \rightarrow \mathcal{A}_X^0 \rightarrow K^1 \rightarrow 0, \\ 0 \rightarrow K^1 \rightarrow \mathcal{A}_X^1 \rightarrow K^2 \rightarrow 0. \end{aligned}$$

The boundary map gives:

$$H_{dR}^2(X) = \frac{H^0(X, K^2)}{dH^0(X, \mathcal{A}_X^1)} \xrightarrow{\delta'} H^1(X, K^1) \xrightarrow{\alpha'} H^2(X, \mathbb{Z}).$$

If ∇ is a connection on L with local connection forms ω_i on U_i , we have:

$$\begin{aligned} \omega_i &= g_{ij} \cdot \omega_j \cdot g_{ij}^{-1} + dg_{ij} \cdot g_{ij}^{-1}, \\ \omega_j - \omega_i &= -g_{ij}^{-1} dg_{ij} = -d \log g_{ij}. \end{aligned}$$

Thus, putting it all together, we have

$$\delta^1\{\omega_i\} = \{\omega_j - \omega_i\} = \{-d \log g_{ij}\} = -2\pi i \delta^0 \left\{ \frac{1}{2\pi i} \log g_{ij} \right\} = -2\pi i c_1(L).$$

□

A first consequence of this theorem is that the image of $c_1^{\mathbb{R}} : \text{Pic}(X) \rightarrow H^2(X, \mathbb{R})$ lies in the $H^{1,1}$ -component.

This gives a necessary condition for a line bundle to be holomorphic. Let us prove that it is sufficient.

Proposition 7.34. *Let $\beta \in H^{1,1}(X) \subseteq H^2(X, \mathbb{R})$ denote a complex line bundle L . Then L admits a holomorphic structure.*

Proof. Let ∇ be a connection on L . By Theorem 7.33, we know that $[\frac{1}{2\pi i} F_{\nabla}] = \beta \in H^{1,1}(X, \mathbb{R})$. Thus, there exists a closed real $(1,1)$ -form ζ such that $[\zeta] = \alpha = [\frac{1}{2\pi i} F_{\nabla}]$.

Since $[\zeta - \frac{1}{2\pi i} F_{\nabla}] = 0$, there exists α such that $d\alpha = \zeta - \frac{1}{2\pi i} F_{\nabla}$. Consider the modified connection $\tilde{\nabla} = \nabla + 2\pi i \alpha$. Then $F_{\tilde{\nabla}} = F_{\nabla} + 2\pi i d\alpha = \zeta \in \mathcal{A}_X^{1,1}$.

Thus $\tilde{\nabla}$ is a compatible connection, and so L admits a holomorphic structure by Theorem 5.7. □

Let us combine Theorem 7.33 with the line bundle–divisor correspondence. For that, we need to recall the following Poincaré map.

Let $\iota : Y \rightarrow X$ be a smooth hypersurface. The *Poincaré map* $\eta_Y \in H_{dR}^{2n-2}(X, \mathbb{R})^* \cong H_{dR}^2(X, \mathbb{R})$ is given by

$$\begin{aligned} \eta_Y : H_{dR}^{2n-2}(X, \mathbb{R}) &\rightarrow \mathbb{R} \\ \gamma &\mapsto \langle \iota^*(\gamma), [Y] \rangle . \end{aligned}$$

This extends to a well-defined map

$$\begin{aligned} \eta : \text{Div}(X) &\rightarrow H_{dR}^2(X, \mathbb{R}) \\ \sum_i a_i [Y_i] &\mapsto \sum_i a_i \eta_{Y_i} , \end{aligned}$$

since X is compact. We have

Theorem 7.35. *Let $L = \mathcal{O}(D)$ for some divisor $D \in \text{Div}(X)$. Then $c_1(L) \in H_{dR}^2(X)$ is the image of D under the Poincaré map.*

Proof. Since c_1 is a linear map, we may assume $D = [Y]$ is an irreducible hypersurface. Choose h a metric on $L = \mathcal{O}(Y)$ and let ∇ be its Chern connection, with curvature F_∇ . The claim of the theorem is equivalent to

$$\int_X F_\nabla \wedge \gamma = -2\pi i \int_Y \iota^*(\gamma) ,$$

for all $\gamma \in \Omega_{closed}^{2n-2}(X)$.

[ADD LATER]

□

8 Kähler Manifolds

We now move on to discuss an important class of complex manifolds: Kähler manifolds.

The idea behind Kähler manifolds is to have a compatible metric with the (almost) complex structure, not just as a hermitian structure, but also satisfies some differential constraints, similar to the vanishing of the Nijenhuis tensor we saw in Section 2.2. In fact, we have

Definition 8.1. Let (X, g, J) be an almost hermitian manifold, and let ∇ denote the Levi-Civita connection of g . We say g is a *Kähler metric* if $\nabla J = 0$. A manifold equipped with a Kähler metric is called a *Kähler manifold*.

The following proposition gives a more hands-on approach to Kähler metrics.

Proposition 8.2. *An almost hermitian manifold (X, g, J) is Kähler if and only if*

- (i) *its Nijenhuis tensor vanishes $N_J = 0$, and*
- (ii) *the associated $(1, 1)$ form ω is closed, $d\omega = 0$.*

Sketch. First notice that $\nabla J = 0$ is equivalent to $\nabla \omega = 0$. Now, one can split $\nabla \omega$ into its symmetric and antisymmetric parts, $\nabla \omega = (\nabla \omega)^s + (\nabla \omega)^{as}$.

Since the Levi-Civita connection is torsion-free, $(\nabla \omega)^{as} = d\omega$. With some work, one can identify $(\nabla \omega)^s$ with the Nijenhuis tensor. □

We refer the interested reader to the seminal paper of Gray and Hervella [GH80] for a detailed proof, as well as an extended discussion around the topic of Kähler metrics and their generalisations. By the proposition above, the defining $(1, 1)$ -forms ω is a symplectic form. Thus, one can view a Kähler manifold as a complex manifold carrying a compatible symplectic structure.

Proposition 8.3. *Let (X, g, J) be a hermitian manifold. Then g is a Kähler metric if and only if the Levi-Civita and the Chern connections coincide.*

Proof. The necessity is clear. Sufficiency is proved in detail in [Huy05, Prop. 4. A.7]. □

9 Positivity and vanishing

10 The Kodaira embedding theorem

11 Kodaira-Spencer deformation theory

12 The Tian-Todorov theorem

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