Lecture Notes on Complex Geometry

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Contents

This course aims to give an introduction to the world of complex geometry. The main idea I would like to convey to the reader is the strong local-to-global properties that holomorphic functions possess, and thus manifolds whose transition functions are holomorphic: complex manifolds.
I have based these notes on the two excellent books, the first by Daniel Huybrechts [Huy05] and the other by Jean-Pierre Demailly, [Dem12], who unfortunately passed away before the book was ever published, and only online drafts are available.

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1 Holomorphic functions

We begin by reviewing some general properties of holomorphic functions and their extension to arbitrary dimensions. We identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$ as real vector spaces. Recall

Definition 1.1. A function $f : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$ is *differentiable at* 0 if its differential at $z_0 \in \mathbb{R}^{2n}$ exists. Equivalently there exists a linear map Df_{z_0} such that

$$f(z) = f(z_0) - Df_{z_0}(z - z_0) + O((z - z_0)^2).$$

Definition 1.2. Let $f : \mathbb{C}^n \to \mathbb{C}^m$ a function. We say f is *holomorphic* at z_0 if it is differentiable, and its differential Df_{z_0} is complex-linear, $Df_{z_0} \in \mathrm{GL}(n, \mathbb{C}) \subseteq \mathrm{GL}(2n, \mathbb{R})$.

The complex-linear condition translates to the equation

$$JDf_{z_0} = Df_{z_0}\, J, \tag{1}$$

where J is the standard complex structure matrix on $\mathbb{R}^{2n}$ (complex multiplication by i in $\mathbb{C}^n$). Equation (1) is the famous *Cauchy–Riemann equations* in a coordinate-free form.
If we take standard coordinates $z_1 = x_1 + iy_1, \dots, z_n x_n + iy_n$, and writing f as $f = u + iv$, the Cauchy–Riemann(CR) equations read

$$\begin{cases} \partial_{x_i} u = \partial_{y_i} v, \\ \partial_{y_i} u = -\partial_{x_i} v, \end{cases} \qquad i = 1, \dots, n\ ,$$

which is perhaps a more standard presentation of the Cauchy–Riemann equations. We give one final incarnation of the Cauchy–Riemann equations, but for that, we need the Wirtinger operators:

$$\frac{\partial}{\partial z_i} := \frac{1}{2}\Big(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}\Big), \qquad \frac{\partial}{\partial \overline{z_i}} := \frac{1}{2}\Big(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i}\Big)\ .$$

Lemma 1.3. *The Wirtinger operators satisfy*

(i) $\displaystyle\frac{\partial}{\partial z_i} f = \overline{\left(\frac{\partial}{\partial \overline{z_i}} \overline{f}\right)}$,

(ii) $\displaystyle\frac{\partial}{\partial z_i}(z_i) = 1,$ *and* $\displaystyle\frac{\partial}{\partial z_i}(\overline{z_i}) = 0,$

(iii) (Chain rule)

$$\frac{\partial}{\partial z_i}(f \circ g) = \sum_{k=1}^{n} \frac{\partial f}{\partial w_k}\frac{\partial g_k}{\partial z_i} + \frac{\partial f}{\partial \overline{w_k}}\frac{\partial \overline{g_k}}{\partial z_i} \qquad \frac{\partial}{\partial \overline{z_i}}(f \circ g) = \sum_{k=1}^{n} \frac{\partial f}{\partial w_k}\frac{\partial g_k}{\partial \overline{z_i}} + \frac{\partial f}{\partial \overline{w_k}}\overline{\left(\frac{\partial g_k}{\partial z_i}\right)}\ .$$

Moreover, the Cauchy–Riemann equations are equivalent to

$$\frac{\partial f}{\partial \overline{z_i}} = 0 \qquad i = 1, \dots n\ ,$$

Proof. Linear algebra exercise. □

We can consider the complexified derivative

$$Df(z_0)^{\mathbb{C}} : T_{z_0}\mathbb{R}^{2n} \otimes \mathbb{C} \longrightarrow T_{f(z_0)}\mathbb{R}^2 \otimes \mathbb{C}.$$

The space $T_{z_0}\mathbb{R}^{2n}\otimes\mathbb{C}$ (resp. $T_{f(z_0)}\mathbb{R}^n\otimes\mathbb{C}$) admits the canonical coordinate base $\{\partial/\partial z_i, \partial/\partial\overline{z_i}\}$ (resp. $\{\partial/\partial w, \partial/\partial\overline{w}\}$). In this base, the Jacobian in block form takes the form
The a holomorphic map f, the matrix of derivatives has the form

$$Df = \begin{pmatrix} \frac{\partial f}{\partial z_i} & 0 \\ 0 & \overline{\frac{\partial f}{\partial z_i}} \end{pmatrix} ,$$

reflecting complex-linearity (no $\partial/\partial\bar{z}$-components) of f. It follows that for any holomorphic function f, $\det\left(Df(z_0)^{\mathbb{C}}\right)$ is real and non-negative; $\det\left(Df(z_0)\right) \geq 0$.

Definition 1.4. A holomorphic map $f : U \to V$ is called *biholomorphic* if there exists a holomorphic inverse g to f.

If f is holomorphic and regular (non-degenerate Jacobian), then its Jacobian determinant satisfies

$$\det Df = \left|\det\left(\frac{\partial f}{\partial z_i}\right)\right|^2 > 0.$$

In particular, $\det(Df) \neq 0$ is the local invertibility criterion. Indeed, we have the holomorphic version of the inverse function theorem:

Theorem 1.5 (Holomorphic Inverse Function Theorem)**.** *Let $U, V \subseteq \mathbb{C}^n$ open and $f : U \to V$ a holomorphic map. Consider $z_0 \in U$ such that* $\det(Df(z_0) \neq 0$. *Then there exist open subsets $z_0 \in U'cU$ and $f(z_0) \in V'CV$ such that f restricts to a biholomorphism.*

More generally, a holomorphic map $f : U \to V$ is called a regular (submersion/immersion as appropriate) when the complex-linear partials $\{\partial f/\partial z_i\}_{i=1}^n$ are surjective (or injective) as needed.

Theorem 1.6 (Holomorphic Implicit Function Theorem)**.** *Let $U \subseteq \mathbb{C}^n$ and $V \subseteq \mathbb{C}^m$ be open sets with $n > m$ and $f : U \to V$ a holomorphic function. Assume that there is z_0 such that $Df(z_0)$ satisfies*

$$\det\left[\left(\frac{\partial f_i}{\partial z_j}\right)_{i,j=1,\ldots n}\right] \neq 0 \ . \tag{2}$$

Then there exists open sets $U_1 \subseteq \mathbb{C}^{n-m}$, $U_2 \subseteq \mathbb{C}^m$ such that $U_1 \times U_2 \subseteq U$ and a holomorphic function $g : U_1 \to U_2$ satisfying $f(w, g(w)) = f(z_0)$ for all $w \in U_1$.

Proof. The inverse function theorem guarantees the existence and differentiability of g. We need to show that g is holomorphic. Indeed, by the chain rule of Lemma 1.3, we have

$$0 = \frac{\partial}{\partial \overline{w_j}}\Big[f_i(w, g(w))\Big] = \frac{\partial f_i}{\partial \overline{w_j}} + \sum_{k=1}^{n} \frac{\partial f_i}{\partial z_k}\frac{\partial g_k}{\partial \overline{w_j}} + \frac{\partial f_i}{\partial \overline{z_k}}\overline{\left(\frac{\partial g_k}{\partial w_j}\right)} = \sum_{k=1}^{n} \frac{\partial f_i}{\partial z_k}\frac{\partial g_k}{\partial \overline{w_j}} \ ,$$

where the first and third terms in the middle line vanish since f is holomorphic.
But the condition in Equation (2) implies that $\left(\frac{\partial f_i}{\partial z_j}\right)$ is invertible, so the only way the second line can vanish is if $\frac{\partial g}{\partial z_j} = 0$, as needed. □

A straightforward corollary of the Holomorphic Implicit Function Theorem is the existence of left (resp. right) holomorphic inverses. We have

Corollary 1.7. *Let $U \subseteq \mathbb{C}^n$ and $V \subseteq \mathbb{C}^m$ be open sets and $f : U \to V$ a holomorphic function. Assume we have $z_0 \in U$ such that $Df(z_0)$ has maximal rank. Then,*

(i) If $n > m$, there exists open sets $z_0 \in U' \subset U$ and $V' \subseteq V$, and a biholomorphic map $g : V' \to U'$ such that $f \circ g = \mathrm{Id}$ in V'.

(ii) If $n < m$, there exists open sets $U' \subset U$ and $f(z_0) \in V' \subseteq V$, and a biholomorphic map $g : V' \to U'$ such that $g \circ f = (\mathrm{Id}_n, 0)$ in U'.

1.1 Cauchy Integral Formula and power series expansion

Recall that a key result of complex analysis is the integral formula of Cauchy:

Theorem 1.8 (Cauchy Integral Formula)**.** *Let $K \subseteq \mathbb{C}$ be a compact subset with piecewise C^1 boundary $C = \partial K$, and $f : K \to \mathbb{C}$ a differentiable function. Then for $w \in K \setminus \partial K$, we have*

$$2\pi i f(w) = \int_C \frac{f(z,\overline{z})}{z-w}dz + \int_K \frac{\partial f}{\partial \overline{z}}\frac{dz \wedge d\overline{z}}{z-w} \tag{3}$$

Proof. Without loss of generality, we assume $w = 0$. We want to study the function $f(z,\overline{z})/z \in L^1(K)$. Taking $\delta > 0$, we have on one side,

$$\int_{K\setminus B_\delta(0)} d\left(\frac{f(z,\overline{z})}{z}\right) dz = -\int_{K\setminus B_\delta(0)} \frac{\partial f}{\partial \overline{z}}\frac{dz \wedge d\overline{z}}{z} \ .$$

On the other side, by Stokes' theorem, we get

$$\int_{K\setminus B_\delta(0)} d\left(\frac{f(z,\overline{z})}{z}\right) dz = \int_C \frac{f(z,\overline{z})}{z}dz - \int_{\partial B_\delta} \frac{f(z,\overline{z})}{z}dz \ .$$

Parametrising the last term in polar coordinates $z = \delta e^{i\theta}$, we have

$$\int_{\partial B_\delta} \frac{f(z,\overline{z})}{z} = \int_0^{2\pi} f(\delta,\theta) i d\theta \ .$$

Putting everything together and taking δ to zero, the claim follows by continuity of f. □

Of course, we are mostly interested in the case where f is holomorphic, so the last term in (3) vanishes, and we have the usual expression

$$f(w) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-w} dz \tag{4}$$

The Cauchy Integral Formula (CIF) generalises to higher dimensions by considering polydiscs $D_R(w) = B_{R_1}(w_1) \times \ldots B_{R_n}(w_n)$ and iterative use of Fubini's theorem.

Exercise 1.9. *Prove the n-dimensional Cauchy Integral Formula in detail:*

$$f(w) = \frac{1}{(2\pi i)^n} \int_{\partial D_R(w)} \frac{f(z_1, \ldots, z_n)}{(z_1 - w_1) \ldots (z_n - w_n)} dz_1 \ldots dz_n \ .$$

The CIF has some important, remarkable consequences for the regularity of the function f:

Proposition 1.10. *Let $f : U \to \mathbb{C}$ be a holomorphic function. Then f is analytic. That is, it admits a convergent power series expansion*

$$2\pi i f(z) = \sum_{|\alpha| \geq 0} \frac{f^{(\alpha)}(z_0)}{\alpha!} z^\alpha \ ,$$

with α a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, .

Proof. We argue the case $n = 1$; the higher-dimensional case follows. We know $\frac{1}{z-w} = \frac{1}{z} \frac{1}{(1 - w/z)} = \sum_{k \geq 0} \frac{w^k}{z^{k+1}}$ for $|w| < |z|$. Substituting in the CIF and using Lebesgue monotone convergence, we have

$$2\pi i f(w) = \int_C \frac{f(z)}{z-w} dz = \int_C \sum_{k \geq 0} w^k \frac{f(z)}{z^{k+1}} dz = \sum_{k \geq 0} w^k \int_C \frac{f(z)}{z^{k+1}} dz \ .$$

Analyticity follows. The coefficients of the power expansion are the successive derivatives of f by the uniqueness of Taylor expansions. Alternatively, one can check directly:

$$\begin{aligned} f'(w) &= \lim_{h \to 0} \frac{f(w+h) - f(w)}{h} = \lim_{h \to 0} \frac{1}{2\pi i h} \int_C \frac{f(z)}{z - (w+h)} - \frac{f(z)}{z-w} dz \\ &= \lim_{h \to 0} \frac{1}{2\pi i h} \int_C \frac{h f(z)}{(z-w-h)(z-w)} dz = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-w)^2} dz \ . \end{aligned}$$

□

The analyticity of holomorphic functions has some remarkable consequences:

Theorem 1.11 (Open mapping theorem)**.** *Let $f : U \to \mathbb{C}$ be a non-constant holomorphic function on an open set U. The f is an open mapping.*
In particular, if there exists $z_0 \in U$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in U$, f is constant.

Theorem 1.12 (Identity principle)**.** *Let U be an open connected subset of $\mathbb{C}^n$ and $f, g : U \to \mathbb{C}$ holomorphic functions. If $f = g$ on an open subset $V \subset U$, then $f \equiv g$ on all of U.*

Proof. Let

$$W = \left\{ z \in U \;\middle|\; \frac{\partial^\alpha f}{\partial z^\alpha} = \frac{\partial^\alpha g}{\partial z^\alpha} \quad \forall\, \alpha \;\; \text{multi-index} \right\} .$$

The set W is clearly closed and non-empty. By analyticity, W is also open, and by connectedness, $W = U$. □

Another consequence of the Cauchy Integral Formula, Equation (4), is

Lemma 1.13 (Cauchy inequality)**.** *Let $f : U \to \mathbb{C}$ be a holomorphic function and take $R > 0$ such that the ball $B_R(z_0)$ is contained in U. Then*

$$|f^{(\alpha)}(z_0)| \leq \frac{\alpha!}{R^\alpha} \sup_{\partial B_R(z_0)} |f(z)| \tag{5}$$

There are two important corollaries of this inequality:

Theorem 1.14 (Generalised Liouville theorem)**.** *Let $f : \mathbb{C}^n \to \mathbb{C}$ a holomorphic function such that $|f(z)| \leq C(1 + |z|)^D$ for some $C, D \geq 0$. Then f is a polynomial with degree at most D.*

Theorem 1.15 (Montel's theorem)**.** *Let $U \subseteq \mathbb{C}^n$ open, and consider $\mathcal{O}(U)$ the space of holomorphic functions on U, equipped with the uniform convergence on compact sets topology, induced by $C^0(U)$. Then every locally uniformly bounded sequence $(f_j)_j \subseteq \mathcal{O}(U)$ has a convergent subsequence.*

Proof. By Arzelà–Ascoli. □

1.2 Hartogs' phenomenon and Weierstrass Preparation theorem

So far, all properties that we have discussed are direct analogues of properties that occur in complex analysis ($n = 1$) and have discussed the rigidity of holomorphic functions. First, we need the following technical lemma

Lemma 1.16. *Consider the open cylinder $U \times V$ with $U \subseteq \mathbb{C}^n$ open, and $V \subseteq \mathbb{C}$ a neighbourhood of $\partial B_\varepsilon(z_0)$ and let $f : V \times U \to \mathbb{C}$ a holomorphic function. Then*

$$g(z_1, \ldots, z_n) := \int_{\partial B_\varepsilon(z_0)} f(\xi, z_1, \ldots z_n) d\xi$$

is a holomorphic function on U.

Proof. Notice that if f were holomorphic on $U \times B_\varepsilon(z_0)$, we would essentially be done. The idea is to reduce it to an equivalent situation.
Since $\partial B_\varepsilon(z_0)$ is compact, for every $\delta > 0$, there exists finitely many ξ_i such that $\{B_\delta(\xi_i)\}$ cover $\partial B_\varepsilon(z_0)$. By choosing δ small enough, we can ensure $B_\delta(\xi_i) \subseteq V$ and f has a convergent power series in $B_\delta(\xi_i) \times U_i$ for all i.
We can now split the integral into a finite sum of integrals where f has a power series expansion. □

Let us now focus on the extension problem.

Theorem 1.17 (Hartogs' principle)**.** *Let $D_R(0)$ and $D_{R'}(0)$ be two polydiscs in $\mathbb{C}^n$ with $\overline{D_{R'}(0)} \subseteq D_R(0)$ so $R_i > R'_i$ for all i. Any holomorphic function $f : D_R(0) \setminus \overline{D_{R'}(0)} \to \mathbb{C}$ can be uniquely extended to a holomorphic function $\overline{f} : D_R(0) \to \mathbb{C}$.*

Proof. Let $w = (z_2, \ldots, z_n)$ with $|z_2| > R'_2$. We can use the Cauchy formula for the function $z \mapsto f(z, w)$, for $R'_1 < \delta < R_1$:

$$f(z, w) = \frac{1}{2\pi i} \int_{|\xi|=\delta} \frac{f(\xi, w)}{(\xi - z)} \, d\xi$$

The integrand is $(\xi, z, w) \mapsto \frac{f(\xi,w)}{(\xi-z)} \, d\xi$, which is holomorphic on $B_c(\delta) \times B_{\delta-c}(0) \times D_{R_2,\ldots,\ R_n}(0)$ for some small c. Therefore, by the lemma, the function

$$\tilde{f}(z, w) = \frac{1}{2\pi i} \int_{|\xi|=\delta} \frac{f(\xi, w)}{(\xi - z)} \, d\xi$$

is holomorphic on $B_{\delta-c}(0) \times D_{R_2,\ldots,R_n}$, providing the desired extension by the identity principle. □

We conclude this subsection by proving two technical lemmas, due to Weierstrass, that will be useful throughout the course.

Definition 1.18 (Weierstrass Polynomial)**.** A *Weierstrass polynomial* in z_1 of degree d is a polynomial

$$z_1^d + a_1(z')z_1^{d-1} + \cdots + a_d(z'),$$

where $a_i(z')$ are holomorphic functions in $z' = (z_2, \ldots, z_n)$ defined in a neighbourhood of the origin and such that $a_i(0, \ldots, 0) = 0$.

Theorem 1.19 (Weierstrass Preparation Theorem)**.** *Let $f : D_\varepsilon(0) \to \mathbb{C}$ with $f(0, 0) = 0$ and $f(z_1, 0, \ldots, 0) \not\equiv 0$. Then for some smaller ball $D_{\varepsilon'}(0)$ there exists a unique decomposition:*

$$f = g \cdot h$$

where g is a Weierstrass polynomial in z_1, and $h : D_{\varepsilon'}(0) \to \mathbb{C}$ is a holomorphic function without zeroes.

Proof. By taking ε smaller if needed, we may assume $f(z_1, 0, \ldots, 0)$ vanishes only at 0, with multiplicity d.
For small w, the zeros of $f_w(z) = f(z, w)$ are given by $a_1(w), \ldots, a_d(w)$. Define:

$$g(z, w) = \prod_{i=1}^{d} (z_1 - a_i(w)), \quad h = \frac{f}{g}$$

We need to show that g and h are holomorphic in z_1 and w. Holomorphicity in z_1 is straightforward.
To see g is holomorphic in w, notice that this amounts to showing that the elementary symmetric polynomials in terms of $a_i(w)$ are holomorphic, which are linear combinations of $S_k = \sum_{i=1}^{n} a_i(w)^k$ for $k = 0, \ldots, d$. By the Cauchy residue formula, we have

$$\sum_{i=1}^{n} a_i(w)^k = \frac{1}{2\pi i} \int_{|\xi|=\varepsilon_1} \frac{\xi^k}{f(\xi, w)} \frac{\partial}{\partial \xi} f(\xi, w) d\xi \ ,$$

which is holomorphic by Lemma 1.16. Finally, we may write

$$h(z_1, w) = \frac{1}{2\pi i} \int_{|\xi|=\varepsilon_1'} \frac{h(\xi, w)}{\xi - z_1} d\xi,$$

which is everywhere holomorphic by Lemma 1.16 and f/g being holomorphic on the annulus.
□

1.3 The ring of holomorphic germs $\mathcal{O}_{\mathbb{C}^n,0}$

We study the local behaviour of holomorphic functions on an arbitrarily small neighbourhood of a point. More formally, this leads to considering the notion of germs and stalks:

Definition 1.20. The *holomorphic stalk at the origin*, denoted $\mathcal{O}_{\mathbb{C}^n,0}$, is the set of all equivalence classes of pairs (U, f), where U is an open neighbourhood of 0 in $\mathbb{C}^n$ and $f : U \to \mathbb{C}$ is a holomorphic function.
Two pairs (U, f) and (V, g) are considered equivalent if there exists an open neighbourhood $W \subseteq U \cap V$ of 0 such that f and g agree on W:

$$(U, f) \sim (V, g) \iff f|_W = g|_W \text{ for some open } W \ni 0.$$

An equivalence class is called a *holomorphic germ at* 0.

Alternatively, one can think of the holomorphic stalk as the set of convergent power series inside $\mathbb{C}[[z_1, ..., z_n]]$.

Exercise 1.21. *Prove that this is indeed the case, i.e. there is a one-to-one correspondence between convergent power series and holomorphic germs.*

Remark 1.22. Definition 1.20 might feel overly complicated and slightly unnatural. Indeed, stalks and germs are better understood in the language of sheaves, which we will introduce in Section 3.

The holomorphic stalk $\mathcal{O}_{\mathbb{C}^n,0}$ inherits a ring structure from that of holomorphic functions. We devote ourselves to studying its structure. We shall prove

Theorem 1.23. *The stalk of holomorphic germs $\mathcal{O}_{\mathbb{C}^n,0}$ is*

(i) a local integral domain,

(ii) a unique factorisation domain (UFD), and

(iii) Noetherian.

It is clear that $\mathcal{O}_{\mathbb{C}^n,0}$ is a local integral domain with the maximal ideal $\mathcal{I}_0$ given by (germs of) functions vanishing at the origin, and residue field $\mathcal{O}_{\mathbb{C}^n,0}/\mathcal{I}_0 \cong \mathbb{C}$.

Proof of Theorem 1.23 (ii).
We prove this by induction. The case $n = 0$ is trivial.
Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$ vanishing at the origin. By the Weierstrass preparation theorem, we can uniquely write f as $f = u \cdot p$, with $u \in \mathcal{O}^{\times}_{\mathbb{C}^n,0}$ a unit and $p \in \mathcal{O}_{\mathbb{C}^{n-1},0}[w]$ (the germ of) a Weierstrass polynomial.
The $\mathcal{O}_{\mathbb{C}^{n-1},0}$ is a UFD by induction hypothesis, and so is $\mathcal{O}_{\mathbb{C}^{n-1},0}[w]$ by Gauss' lemma.
It remains to check that p is a finite irreducible element of $\mathcal{O}_{\mathbb{C}^n,0}$, which is straightforward using the uniqueness of the decomposition of the Weierstrass Preparation Theorem ☐

Let us now prove that the $\mathcal{O}^{\times}_{\mathbb{C}^n,0}$ is Noetherian, that is, every ideal is finitely generated. First, we need another technical lemma of Weierstrass:

Theorem 1.24 (Weierstrass Division Theorem)**.** *Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$, and let g be a Weierstrass polynomial of degree d. Then there exist a unique $h \in \mathcal{O}_{\mathbb{C}^n,0}$ and $r \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ with* $\deg r < d$ *such that:*

$$f = g \cdot h + r$$

Proof. Define

$$h(z, w) = \frac{1}{2\pi i} \int_{\partial B_\varepsilon(0)} \frac{f(\xi, w)}{g(\xi, w)} \frac{d\xi}{\xi - z}$$

and check that $r = f - gh$ lies in $\mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ and is of degree $< d$ holomorphicity. ☐

Proof of Theorem 1.23 (iii). Again, we prove this by induction, with the case $n = 0$ being immediate.
Assume $\mathcal{O}_{\mathbb{C}^{n-1},0}$ is Noetherian, and therefore so is the subring $\mathcal{O}_{\mathbb{C}^{n-1},0}[z_1] \subseteq \mathcal{O}_{\mathbb{C}^n,0}$, by Hilbert's basis theorem.

Let $I \in \mathcal{O}_{\mathbb{C}^n,0}$ an ideal, so $I \cap \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ is finitely generated.
Take $f \in I$.By the Weierstrass Preparation Theorem, we get $f = gh$ with $h \in \mathcal{O}^*_{\mathbb{C}^n,0}$ and $g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$, so $g = fh^{-1} \in I \cap \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$.
For any other $\tilde{f} \in I$, the Weierstrass division theorem implies that $\tilde{f} = g\tilde{h} + \tilde{r}$ for $r \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$. Since $\tilde{f}$ and g are in I, it follows that $r \in I \cap \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$. Thus, I is a finitely generated ideal. □

1.4 Hilbert's Nullstellensatz

In the previous section, we studied holomorphic functions and their germs. The goal of this section is to relate them to a more geometric notion, namely analytic sets and their germs. Given $f : U \to \mathbb{C}$ a holomorphic function, we denote its vanishing set as $Z(f) = \{z \in U \big| \, f(z) = 0\}$.

Definition 1.25. An *analytic set* $Z \subseteq X$ is a set such that for each $x \in Z$, there exists an open neighbourhood $U \ni x$ and holomorphic functions $f_1, \dots, f_k \in \mathcal{O}(U)$ with $Z \cap U = Z(f_1, \dots, f_k) = \bigcap_{i=1}^{k} Z(f_i)$.

In the same spirit as before, we define the corresponding germs

Definition 1.26. An *analytic germ* at $x \in X$ is an equivalence class of analytic sets under the relation $Z_1 \sim Z_2$ if $Z_1 \cap U = Z_2 \cap U$ for some neighbourhood $U \ni x$.

Given a germ X at the origin, we denote by $I(X)$ the set of homomorphic germs s satisfying the condition $X \subseteq Z(s)$. So $Z(\cdot)$ takes holomorphic germs (or functions) to analytic germs, and $I(\cdot)$ takes analytic germs to their holomorphic counterparts. They satisfy the following relations:

Lemma 1.27.

(i) For any subset $A \subseteq \mathcal{O}_{X,x}$, $Z(A)$ is a well-defined analytic germ with $Z(A) = Z((A)_{\mathcal{O}_{X,x}})$.

(ii) For every analytic germ Z, $I(Z) = \{f \in \mathcal{O}_{X,x} \mid Z \subset Z(f)\}$ is an ideal.

(iii) If $X_1 \subset X_2$ are analytic germ, then $I(X_2) \subset I(X_1)$. If $I_1 \subset I_2$ are ideals in $\mathcal{O}_{X,x}$, then $Z(I_2) \subset Z(I_1)$.

(iv) $Z = Z(I(Z))$ and $I \subset I(Z(I))$.

(v) $Z(I \cdot J) = Z(I) \cup Z(J)$ and $Z(I + J) = Z(I) \cap Z(J)$.

Proof. Exercise. □

The relation between holomorphic and analytic germs is made precise by Hilbert's Nullstellensatz:

Theorem 1.28 (Hilbert's Nullstellensatz Theorem). *For any ideal $I \subseteq \mathcal{O}_{X,x}$, we have:*

$$\sqrt{I} = I\big(Z(I)\big)$$

where $\sqrt{I}$ is the radical ideal of I; $\sqrt{I} = \{f \in \mathcal{O}_{X,x} \mid f^n \in I$ for some $n\}$.

We would like to understand the fundamental "building blocks" of holomorphic and analytic germs. Since the holomorphic stalk naturally carries a ring structure, our focus will be on its prime ideals. On the side of analytic germs, we introduce the following definition:

Definition 1.29. An analytic germ is Z called *irreducible* if for any union $Z = Z_1 \cup Z_2$ with Z_i analytic germs, either $Z = Z_1$ or $Z = Z_2$.

As expected, we have the following result

Lemma 1.30. *An analytic germ Z is irreducible if and only if $I(Z)$ is a prime ideal.*

Proof. Let $f_1 f_2 \in I$. Then $Z = \big(Z \cap Z(f_1)\big) \cup \big(Z \cap Z(f_2)\big)$. If Z is irreducible $Z = Z \cap Z(f_i)$, so f_i vanishes along Z, i.e. $f_i \in I(Z)$.
The converse follows similarly. □

2 Complex and almost complex manifolds

We now introduce the main class of objects that we are interested in, complex manifolds. We will give two definitions for them. First, using complex charts and holomorphic transition functions. Second, we adopt a more differential geometric style, using $\mathrm{GL}(n, \mathbb{C})$-structures, more commonly known as almost complex structures on a real manifold. The two definitions are equivalent by virtue of the celebrated Newlander-Nirenberg Theorem.
For the remainder of the notes, a (topological) manifold is a locally Euclidean, paracompact, second-countable, Hausdorff space. Recall from differential geometry:

Definition 2.1. A $\mathcal{C}^k$-*manifold* is a manifold equipped with an atlas of charts $(U_i, \phi_i)_{i \in I}$, where transition functions $\phi_{ij} = \phi_i \circ \phi_j^{-1}$ are $\mathcal{C}^k$-diffeomorphisms between open sets in $\mathbb{R}^n$.

Recall that $\mathcal{C}^0$-manifolds are topological manifolds, and that a theorem of Whitney tells us that a $\mathcal{C}^k$-manifold for $k \geq 1$ admits a compatible $\mathcal{C}^\infty$-structure.
Understanding when a manifold admits a smooth structure, and if so, how many, was an active research area in the second half of the 20th century that is nowadays well understood (see e.g. Kervaire–Milnor groups, Kirby–Siebenmann invariants, geometrisation conjecture) except in dimension 4, where surprising links to other areas of mathematics appear.

Another class (before I digress too much) is the class of affine manifolds, where the $\mathcal{C}^k$ condition is replaced by $\mathrm{Aff}(\mathbb{R}^n)$, requiring the transition maps to be affine maps of $\mathbb{R}^n$. Affine manifolds are quite mysterious, and longstanding conjectures and open problems remain to be tackled.

Definition 2.2. A *complex manifold* is a manifold equipped with an atlas of charts $(U_i, \phi_i)_{i\in I}$, where transition functions $\phi_{ij} = \phi_i \circ \phi_j^{-1}$ are biholomorphisms between open sets in $\mathbb{C}^n$.

To avoid issues and pathologies, we will always assume our atlases are maximal, i.e. they are not a proper subset of any other atlas. Every atlas $\{(U_i, \phi_i) : i \in I\}$ is contained in a unique maximal atlas: the set of all charts (U, ϕ) compatible with (U_i, ϕ_i) for all $i \in I$, so there is no prejudice in always taking the maximal atlas.
We will mostly refer to X as the complex manifold, omitting the atlas to lighten notation, as is typically done in differential geometry. As in the previous case, we can ask the questions:

Question 2.3. *When does a manifold M admit the structure of a complex manifold? Is the complex structure unique? Can we classify complex manifolds up to biholomorphism?*

In contrast to the smooth case, very little is known in this case, beyond some obvious topological constraints, discussed in the exercises.
In the compact setting, some existence and classification results exist for complex dimensions 1 and 2. Already in dimension 3, we find one of the most (in)famous open problems in differential geometry:

Question 2.4. *Does the round 6-sphere S^6 admit the structure of a complex structure?*

In the non-compact case, we have Liouville-type obstructions, so we know that the complex plane $\mathbb{C}^n$ is not biholomorphic to certain bounded domains (e.g. the unit ball or polydisc). However, there is no high-dimensional analogue of the Uniformisation Theorem. In general, complex domains carry intrinsic complex-analytic invariants that obstruct biholomorphism. For $n > 1$, many bounded domains are not biholomorphically equivalent.

Definition 2.5. Let X be a complex manifold, and $f : X \to \mathbb{C}$ a function. We call f *holomorphic* if, for all charts (U, ϕ) in the (maximal) atlas, $f \circ \phi$ is a holomorphic function in the sense of Section 1.

Definition 2.6. Let X, Y be complex manifolds and $f : X \to Y$ a continuous function. The map f is said to be holomorphic if for all charts (U, ϕ) of X and (V, ψ) of Y, the map

$$\psi^{-1} \circ f \circ \phi$$

is a holomorphic map in the sense of Section 1.

Definition 2.7. Let X be a complex manifold of dimension n, and $Y \subseteq X$.
We say Y is an (embedded) *complex submanifold* of X of dimension k if for each $y \in Y$ there exist an open neighbourhood U of y and local holomorphic coordinates $(z_1, \ldots, z_n)$ on U such that $Y = Z(z_{k+1}, \ldots, z_n)$.

We will usually require Y to be closed in X. With the definition above, it is easy to see that

Proposition 2.8. *A complex submanifold is a complex manifold such that the inclusion map $\iota_Y : Y \hookrightarrow X$ is injective and holomorphic.*

Conversely, a holomorphic map $f : Y \to X$ is called an embedding if it is injective, locally closed, and with injective differential $Df : T_yY \to T_{f(y)}X$ for all $y \in Y$. It follows easily that f is an embedding if and only if $f(Y)$ is a complex submanifold of X, biholomorphic to Y.
As in the smooth case, we can produce examples of complex submanifolds via the holomorphic implicit function theorem:

Theorem 2.9. *Let $f : X \to Y$ be a holomorphic map between complex manifolds of dimensions n and m respectively, and let $y \in Y$ such that the differential $Df_x : T_xX \to T_yY$ is surjective for all $x \in f^{-1}(y)$.*
Then $f^{-1}(y)$ is a complex submanifold of dimension $n - m$.

A point y satisfying the conditions of the theorem above is called a *regular* point (or value, if $Y = \mathbb{C}$). We have

Corollary 2.10. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function and c a regular value, then $Z(f - c) = f^{-1}(c)$ is a complex hypersurface (complex submanifold) of complex codimension* 1.

Unfortunately, one needs to work a bit harder if one is interested in finding examples of compact complex submanifolds.

Exercise 2.11. *The only compact complex submanifolds of $\mathbb{C}^n$ (when considered as submanifolds of $\mathbb{C}^n$) are discrete points.*

Let us introduce the first compact example, which will play a prominent role throughout the course. The complex projective space $\mathbb{CP}^n$ is the moduli space of complex lines (or dually hyperplanes) in $\mathbb{C}^{n+1}$. It can be realised as the quotient

$$\mathbb{CP}^n \cong (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^* ,$$

where the $\mathbb{C}^*$-action is given by $z \mapsto \lambda z$.
The complex projective space $\mathbb{CP}^n$ is a compact n-dimensional complex manifold.

Let us define homogeneous coordinates $[z_0, \dots, z_n]$ on $\mathbb{CP}^n$. For $i = 0, \dots, n$, define a chart (U_i, ϕ_i) on $\mathbb{CP}^n$ by $U_i = \mathbb{C}^n$ and $\phi_i : \mathbb{C}^n \to \mathbb{CP}^n$ given by

$$\phi_i : (w_1, \dots, w_n) \longmapsto [w_1, \dots, w_i, 1, w_{i+1}, \dots, w_n].$$

This is a homeomorphism with the open subset

$$\phi_i(U_i) = \{[z_0, \dots, z_n] \in \mathbb{CP}^n : z_i \neq 0\} \text{ in } \mathbb{CP}^n.$$

For $0 \leq i < j \leq n$, the transition function $\phi_{ij} = \phi_j^{-1} \circ \phi_i$ is given by

$$\begin{aligned} \phi_{ij} : \mathbb{C}^n \setminus \{z_j = 0\} &\to \mathbb{C}^n \setminus \{z_i = 0\} \\ (z_1, \dots, z_n) &\longmapsto (\frac{z_1}{z_j}, \dots, \frac{z_i}{z_j}, \frac{1}{z_j}, \frac{z_{i+1}}{z_j}, \dots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \dots, \frac{z_n}{z_j}) \ . \end{aligned}$$

The ϕ_{ij}'s are clearly biholomorphisms. So $\{(U_i, \phi_i)\}_{i=0,\dots,n+1}$ forms an atlas of $\mathbb{CP}^n$, that extends to the corresponding maximal atlas.
Now, we have the following example of complex submanifolds:

Proposition 2.12. *Let $p : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}$ a homogeneous polynomial such that 0 is a regular value of p, and consider*

$$X = \{[z_0, \dots, z_n] \in \mathbb{CP}^n \big| \ (z_0, \dots, z_n) \in Z(p)\} \ .$$

Then X is a well-defined compact complex submanifold of $\mathbb{CP}^n$.

Proof. X is well-defined, since p is homogeneous, so $p(z) = 0$ implies $p(\lambda z) = 0$ for all $\lambda \in \mathbb{C}^*$.
Now, X is covered by the charts $V_i = (X \cap U_i)$, where U_i are the standard charts for $\mathbb{CP}^n$ used above. On each V_i, X is described by the vanishing of $p(z_0, \dots, z_{i-1}, 1, z_{i+1}, \dots z_n)$, and Theorem 2.9 concludes the proof. □

We give two examples:

Example 2.13. *For $d \in \mathbb{N}_+$, the set $X = (z_0^d + z_1^d + z_2^d) \subseteq \mathbb{CP}^2$ is a Riemann surface of genus $g = \frac{(d-1)(d-2)}{2}$.*

Example 2.14. *The set $Y = Z(z_0^2 + \cdots + z_3^2) \subseteq \mathbb{CP}^3$ is a projective complex manifold biholomorphic, $\mathbb{CP}^1 \times \mathbb{CP}^1$.*

Of course, one may ask how general the condition for 0 to be a regular value of a homogeneous polynomial. We leave it as an exercise to show that

Exercise 2.15. *The set of homogeneous polynomials for which 0 is a regular value is generic.*

More generally, one has

Proposition 2.16. *Let $(p_1, \dots p_k) : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}^k$ a collection of homogeneous polynomials such that $(0, \dots, 0)$ is a regular value. Then $\big(Z(p_1) \cap \dots \cap Z(p_n)\big)/\mathbb{C}^* \subseteq \mathbb{CP}^n$ is a complex submanifold of dimension $n-k$, called a* complete intersection.

More generally, a projective variety is a subset X of $\mathbb{CP}^n$ which is locally defined by the vanishing of finitely many homogeneous polynomials.
Projective complex manifolds allow us to consider a large number of examples of complex manifolds. Moreover, since they are defined using polynomials, they can be studied using algebraic techniques, giving rise to complex algebraic geometry.
In the opposite direction, one may consider under what conditions one can guarantee that a compact complex manifold X can be realised as a projective complex manifold. The answer to this question is fully understood and follows from two important results, Chow's Theorem and the Kodaira Embedding Theorem, which we will prove during this course.
Complex Lie groups also provide important examples of complex manifolds:

Definition 2.17. A *complex Lie group* is a group G that is also a complex manifold such that multiplication and inversion are holomorphic maps.

Examples include the general linear groups $\mathrm{GL}_n(\mathbb{C})$, special linear groups $\mathrm{SL}_n(\mathbb{C})$, complex tori, etc.

Proposition 2.18. *Let G be a complex Lie group acting holomorphically on a complex manifold X. If the action is free and proper, then the quotient X/G carries a canonical complex manifold structure for which the projection $X \to X/G$ is a holomorphic submersion.*

Proof. □

As a direct application of this proposition, we give two further examples of complex manifolds: Hopf and Iwasawa manifolds.
Hopf manifolds are examples of compact complex manifolds obtained as quotients of $\mathbb{C}^n \setminus \{0\}$ by a discrete group generated by contractions. For a concrete example, let $\alpha \in (0,1)$ and

$$H_A = (\mathbb{C}^n \setminus \{0\})/\sim_\alpha$$

where $z \sim_\alpha w$ if $z = \alpha^n w$ for some n.

Remark 2.19. Hopf manifolds are diffeomorphic to $S^{2n-1} \times S^1$ (think in polar coordinates) and provide important examples in complex geometry, as we shall see.

Finally consider $\mathbb{U} \subseteq \mathrm{GL}(3, \mathbb{C})$ the subgroup of upper-triangular matrices

$$U = \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix}$$

and its subgroup $\mathbb{U}_{\mathbb{Z}} = \mathbb{U} \cap \mathrm{GL}(3, \mathbb{Z}[i])$. The group $\mathbb{U}_{\mathbb{Z}}$ acts by translations $(w_1, w_2, w_3) \cdot (z_1, z_2, z_3) \mapsto (z_1 + w_1, z_2 + w_2, z_3 + w_3)$, which is a free and proper action, so the quotient is a complex manifold, known as the Iwasawa manifold $\mathbb{I} = \mathbb{U}/\mathbb{U}_{\mathbb{Z}}$.
The first and third coordinate provide a holomorphic submersion $f : \mathbb{I} \to \mathbb{C}/\mathbb{Z}[i] \times \mathbb{C}/\mathbb{Z}[i]$, with the fibres given by the remaining coordinate, biholomorphic to $\mathbb{C}/\mathbb{Z}[i]$.

2.1 Almost complex structures

We now introduce the second definition of complex manifolds, via almost complex structures. The idea is to consider a weaker notion of complex structures and study the relation between the two.
The idea is the following: Let X be a complex n-manifold in the sense of Definition 2.2. Then, the underlying topological manifold carries a natural smooth real $2n$-manifold $X_{\mathbb{R}}$. Its tangent bundle $TX_{\mathbb{R}}$ inherits the structure of a complex vector bundle, which is reflected in the existence of a bundle endomorphism $J \in \mathcal{C}^{\infty}(\mathrm{End}(TX_{\mathbb{R}})$ such that $J^2 = -\mathrm{Id}_{2n}$ fiberwise. This motivates the notion of an almost complex structure:

Definition 2.20. Let X be a real $2n$-manifold. An *almost complex structure* J on X is the choice of a section J in $\mathcal{C}^{\infty}(\mathrm{End}(TX_{\mathbb{R}}))$ satisfying the condition $J^2 = -\mathrm{Id}_{2n}$.
A manifold X equipped with an almost complex structure J is called an *almost complex manifold*.

Any complex manifold in the sense of Definition 2.2 induces a real manifold X with an almost complex structure J. The converse is not true, as we shall see.
Since an almost complex structure J furnishes the tangent space with the structure of a complex vector space pointwise, we can define the analogue notions of holomorphic functions and maps.

Definition 2.21. Let (X, J) be an almost complex manifold and $f : X \to \mathbb{C}$ a smooth function. We say f is *J-holomorphic* function if

$$df \circ J = idf \ .$$

Similarly, we have

Definition 2.22. Let (X, I) and (Y, J) be almost complex manifolds and $f : X \to Y$ a smooth map. We say f is a *pseudo-holomorphic* map if

$$df \circ I = J \circ df \ .$$

Before proceeding, let us say a few words about the existence of almost complex structures. Unlike the case of complex strucutres, we are not requiring that our structure solves any PDEs (the transition maps being holomorphic), just the existence of a special section of the endomorphism bundle $\mathrm{End}(TM)$ (or the reduction of the frame bundle to a principal $\mathrm{GL}(n,\mathbb{C})$-bundle). This problem is well-understood from the point of view of classifying spaces, and it allows us to phrase necessary and sufficient conditions for the existence of an almost complex structure in terms of very explicit topological conditions in low dimensions:

Proposition 2.23. *Let M^{2n} be a closed manifold*

(i) For $n = 1$, M admits an almost complex structure if and only if M is orientable (equiv. $w_1(M) = 0$).

(ii) For $n = 2$, M admits an almost complex structure if and only if M is orientable and there exists $h \in H^2(M,\mathbb{Z})$ such that

$$h^2 = 3\sigma(X) + 2\chi(X) \qquad\qquad h \equiv_2 w_2(X)\ .$$

We refer the interested reader to [MS74, §12] for an introductory discussion on obstruction theory on vector bundles.

2.2 The exterior differential and the Nijenhuis tensor

Let us now explore the geometry of almost complex manifolds. For the remainder of the section (X^n, J) will denote an almost complex manifold of (complex) dimension n.

Lemma 2.24. *The vector bundle $TX \otimes_{\mathbb{R}} \mathbb{C}$ splits as a direct sum of complex bundles $TX^{1,0} \oplus TX^{0,1}$ of complex dimension n, given by*

$$TX^{1,0} = \ker\left(i\,\mathrm{Id} - J\right) \qquad\qquad TX^{0,1} = \ker\left(i\,\mathrm{Id} + J\right)$$

Proof. The minimal polynomial of J is $x^2 - 1 = (x-i)(x+i)$, which means J is diagonalisable over $\mathbb{C}$. The bundles $TX^{1,0}$ and $TX^{0,1}$ are the corresponding eigenbundles □

Remark 2.25. While $TX^{1,0}$ and $TX^{0,1}$ are not in general isomorphic as complex bundles, they are always isomorphic as real bundles, with the isomorphism given by conjugation.

The decomposition of the complexified tangent bundle into holomorphic and anti-holomorphic parts trickles down into all associated vector bundles. In particular, we have the following decomposition of exterior k-forms:

$$\Lambda^k T^*M \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q} T^*M \qquad\qquad \Lambda^{p,q}T^*M \coloneqq \Lambda^p\left(T^*X^{1,0}\right) \otimes \Lambda^q\left(T^*X^{1,0}\right)\ .$$

We denote the space of smooth sections of $\Lambda^{p,q}T^*M$ by $\mathcal{A}^{p,q} = \Gamma(X, \Lambda^{p,q}T^*M)$.
We can study how the exterior differential behaves with respect to this decomposition. We have the following:

Proposition 2.26. *There exists operators* $\partial : \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q}$ *and* $\mu : \mathcal{A}^{p,q} \to \mathcal{A}^{p+2,q-1}$ *such that the exterior differential* d *decomposes as*

$$d = \mu + \partial + \overline{\partial} + \overline{\mu} \ ,$$

with $\overline{\partial}$ *and* $\overline{\mu}$ *are the conjugate operators to* ∂ *and* μ *respectively.*

Proof. The exterior differential d is a local operator. Any (p,q)-form γ can be written down locally as

$$\gamma = \sum_{|I|=p, |J|=q} f_{I,J} \ \alpha^I \wedge \overline{\alpha}^J$$

with $\{\alpha_1, \dots, \alpha_n\}$ a local basis of $\mathcal{A}^{1,0}$. □

Lemma 2.27. *The operators* ∂ *and* μ *satisfy the following properties:*

(i) the Leibniz rule,

(ii) ∂ *is* $\mathbb{C}$*-linear and* μ *is function linear, and*

(iii) the following identities hold:

$$\mu\partial + \partial\mu = 0 \ , \qquad \partial^2 + \overline{\partial}\mu + \mu\overline{\partial} = 0 \ ,$$
$$\mu^2 = 0 \ , \qquad \mu\overline{\mu} + \overline{\partial}\partial + \partial\overline{\partial} + \overline{\mu}\mu = 0 \ .$$

Proof. Exercise. □

Since μ is function-linear, we can identify the operator μ acting on $(0,1)$-forms with a tensor $N_J \in \Gamma\big(X, \mathrm{Hom}(T^*X^{0,1}, \Lambda^2 T^*X^{1,0})\big)$ such that $\mu(\alpha) = -N_J(\alpha)$ for $\alpha \in \mathcal{A}^{0,1}$.
The tensor N_J is known as the Nijenhuis tensor and will play a key role in our discussion. Under the canonical identification $\mathrm{Hom}(T^*X^{0,1}, \Lambda^2 T^*X^{1,0}) \cong \Lambda^2 T^*X^{1,0} \otimes TX^{0,1}$, we can view N_J as a skew-symmetric map

$$N_J : TX^{1,0} \times TX^{1,0} \to TX^{0,1} \ .$$

Lemma 2.28. *Under the identification above, the Nijenhuis tensor is given by*

$$N_J(X, Y) = ([X, Y])^{0,1} \ .$$

Proof. Let α be a $(0,1)$-form and X, Y J-holomorphic vector fields. By the definition of μ and N_J, we have that $\big(N_J(\alpha)\big)(X, Y) = -d\alpha(X, Y)$.
Now, we can expand the right-hand side using the usual formula $d\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y])$. The terms $\alpha(X)$ and $\alpha(Y)$ by bidegree reasons, and $\alpha([X,Y])$ only depends on the $(0,1)$-part of the Lie bracket since α is a $(0,1)$-form. □

Exercise 2.29. *The usual definition of the Nijenhuis is*

$$\widetilde{N_J}(X,Y) = [X,Y] + J([JX,Y] + [X,JY]) - [JX,JY] \ .$$

Prove that the two definitions are equivalent (up to complexification and conjugation).

All in all, we have almost proved the following:

Proposition 2.30. *On an almost complex manifold, the following are equivalent:*

(i) $\mu = 0$*,*

(ii) The subbundle $TX^{1,0}$ *is involutive,*

(iii) $\partial^2 = 0$*.*

Proof. The equivalence between (*i*) and (*ii*) follows from Lemma 2.28. Item (*i*) implies (*iii*) by Lemma 2.27. Thus, we only need to show that (*iii*) implies (*ii*).
It suffices to show that $\overline{\partial} f([X,Y]) = 0$ for a function f and $X, Y \in TX^{1,0}$. Now, we have

$$\begin{aligned} 0 = \partial^{\,2} f(X,Y) &= (d\partial f)(X,Y) = X(\partial f(Y)) - Y(\partial f(X)) - \partial f([X,Y]) \\ &= X(df(Y)) - Y(df(X)) - \partial f([X,Y]) = df([X,Y]) - \partial f([X,Y]) \\ &= \overline{\partial} f([X,Y]) \ . \end{aligned}$$

□

An almost complex structure is called integrable if any of the above conditions is satisfied, motivated by the following computation:

Lemma 2.31. *Let* (X,J) *be a complex manifold. Then* $N_J \equiv 0$*.*

Proof. Let $\{z_1, \dots, z_n\}$ be local holomorphic coordinates. Then $\{dz_1, \dots, dz_n\}$ is (pointwise) a basis for $T^*X^{1,0}$. In particular any $\alpha \in \mathcal{A}^{1,0}$ can be locally written as

$$\alpha = \sum_{k=1}^{n} f_k dz_k \ ,$$

In particular, we have

$$d\alpha = \sum_{j,k=1}^{n} \left(\frac{\partial f_k}{\partial z_j} dz_j + \frac{\partial f_k}{\partial \overline{z_j}} dz_j \right) \wedge dz_k \ .$$

□

So the vanishing of the Nijenhuis tensor is a necessary condition for (X,J) to be a complex manifold. In fact, it is also sufficient:

Theorem 2.32 (Newlander–Nirenberg)**.** *An almost complex manifold* (X,J) *admits a compatible complex structure if and only if the almost complex structure* J *is integrable, i.e.* $N_J \equiv 0$

The proof of the Newlander–Nirenberg amounts to constructing local J-holomorphic coordinates. The details of the proof are relatively technical and involved; therefore, we will skip them. You can find a complete proof in [Dem12]
Therefore, one could define a complex manifold as a manifold equipped with an integrable almost complex structure.

Remark 2.33. In fact, one can take a more systematic approach to these questions from the point of view of G-structures. In that framework, the existence of an almost complex structure corresponds to a reduction of the frame bundle to a principal $\mathrm{GL}(n, \mathbb{C})$-bundle, the vanishing of the Nijenhuis tensor corresponds to the structure being 1-integrable, and the Newlander-Nirenberg theorem says that there are no further obstructions from being 1-integrable to being integrable.
We will (hopefully) revisit the world of G-structures when we discuss the Kähler condition in Section 7.

2.3 Cohomologies in complex manifolds

As part of our discussion, we saw that (almost) complex manifolds carry natural operators that square to 0. In particular, this allows us to consider new cohomology theories for these operators.

Remark 2.34. The case of almost complex manifolds is not particularly amenable to having a good cohomology theory since the operator μ is of order 0, so cohomology groups will contain little interesting information. However, one can take this further to produce an interesting cohomology theory, see [CW21].

From now on, we restrict ourselves to the case of complex manifolds. Recall that, since $d^2 = 0$ and $d = \partial + \overline{\partial}$ on a complex manifold, we have $\partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0$. We can define four different cohomology theories on X:

Definition 2.35. Let (X, J) be a complex manifold.

- The *Dolbeault* cohomology

$$H^{p,q}_{\overline{\partial}}(X) = \frac{\ker\left(\overline{\partial} : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(X)\right)}{\operatorname{im}\left(\overline{\partial} : \mathcal{A}^{p,q-1}(X) \to \mathcal{A}^{p,q}(X)\right)}.$$

- The *de Rham* cohomology

$$H^{k}_{dR}(X) = \frac{\ker\left(d : \mathcal{A}^{k}(X) \to \mathcal{A}^{k+1}(X)\right)}{\operatorname{im}\left(d : \mathcal{A}^{k-1}(X) \to \mathcal{A}^{k}(X)\right)}.$$

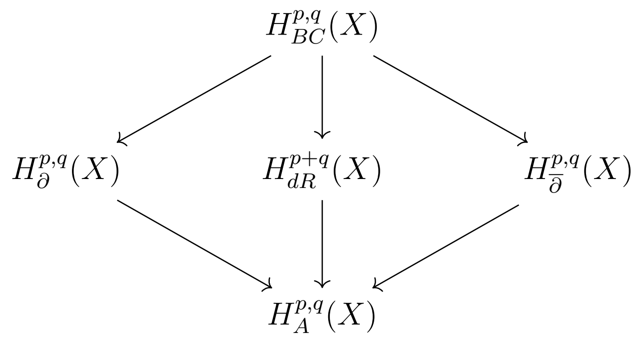
item The *Bott–Chern* cohomology

$$H^{p,q}_{BC}(X) = \left(\frac{\ker\partial \,\cap\, \ker\overline{\partial}}{\operatorname{im}\partial\overline{\partial}}\right)^{p,q}$$

- The *Aeppli* cohomology

$$H_A^{p,q}(X) = \left(\frac{\ker \partial\overline{\partial}}{\operatorname{im}\partial \ + \ \operatorname{im}\overline{\partial}}\right)^{p,q} .$$

These are all well-defined, and there are canonical inclusion maps between the different cohomologies, induced by inclusion and projection:



where $H_{\partial}^{p,q}(X)$ are defined analogously to the Dolbeault cohomology groups, and conjugation yields the isomorphisms $H_{\partial}^{p,q} \cong \overline{H_{\overline{\partial}}^{q,p}}$.
We conclude this section by computing the Dolbeault cohomology groups $H_{\overline{\partial}}^{p,q}$ on a polydisc $D_\varepsilon \subseteq \mathbb{C}^n$, for $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, with $\varepsilon_i = \infty$ allowed. First, we need

Lemma 2.36 (Baby $\overline{\partial}$-Poincaré Lemma)**.** *Let $U \subseteq \mathbb{C}$ be an open set containing the closed ball $\overline{B_\varepsilon}$. For any $\alpha = f d\overline{z} \in \mathcal{A}^{0,1}(U)$, the function*

$$g = \frac{1}{2\pi i}\int_{B_\varepsilon} \frac{f(w)}{w-z} dw \wedge d\overline{w}$$

satisfies $\alpha = \overline{\partial} g$ on B_ε.

Proof. Let us prove that $\alpha = \overline{\partial} g$ in a neighbourhood V of $z_0 \in B_\varepsilon$. Take ψ a bump function such that $\psi|_V \equiv 1$ and $\operatorname{supp}(\psi) \subseteq B_\varepsilon$, and consider the decomposition $f = \psi f + (1-\psi) f =: f_1 + f_2$, and the induced one for g. Let us check that g_1 is a well-defined smooth function. Since f_1 has compact support, we can extend it to the entire complex plane, and by the change of coordinates $w = z + re^{i\phi}$, we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{B_\varepsilon} \frac{f_1(w)}{w-z} dw \wedge d\overline{w} &= \frac{1}{2\pi i}\int_{\mathbb{C}} f(z+re^{i\phi}) \frac{(e^{i\phi}dr + ire^{i\phi}d\phi) \wedge (e^{-i\phi}dr - ire^{-i\phi}d\phi)}{re^{i\phi}} \\ &= \frac{1}{\pi}\int_{\mathbb{C}} f(z+re^{i\phi}) e^{-i\phi} d\phi \wedge dr \ ,\end{aligned}$$

which is clearly smooth in B.

All that remains is to compute $\overline{\partial} g$. Since $\frac{1}{(w-z)}$ is holomorphic in the complement of V, it follows from differentiation under the integral sign that $\overline{\partial} g_2 = 0$. For g_1, using the expression above, we have

$$\begin{aligned}
\overline{\partial} g_1 &= \frac{1}{\pi}\overline{\partial}\int_{\mathbb{C}} f(z+re^{i\phi})e^{-i\phi} d\phi \wedge dr \\
&= \frac{1}{\pi}\int_{\mathbb{C}} \left(\frac{\partial f}{\partial w}\frac{\partial(z+re^{i\phi})}{\partial \overline{z}} + \frac{\partial f}{\partial \overline{w}} \overline{\left(\frac{\partial(z+re^{i\phi})}{\partial z}\right)} \right) e^{-i\phi} d\phi \wedge dr \\
&= \frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial f}{\partial \overline{w}} e^{-i\phi} d\phi \wedge dr = \frac{1}{2\pi i}\int_B \frac{\partial f}{\partial \overline{w}}\frac{dw\wedge d\overline{w}}{w-z} \\
&= f(z) \ ,
\end{aligned}$$

where the second line follows from the chain rule from Lemma 1.3, we undid the change of variables in the third line, and the fourth line follows by the (general) Cauchy Integral Formula, Equation (3). □

By induction on the dimension and bidegree, one shows

Lemma 2.37 ($\overline{\partial}$-Poincaré lemma)**.** *Let $U \subseteq \mathbb{C}^n$ be an open set containing the closed polydisc $\overline{D_\varepsilon}$. For $q > 0$, if $\alpha \in \mathcal{A}^{p,q}(U)$ is $\overline{\partial}$-closed, there exists $\beta \in \mathcal{A}^{p,q-1}(D_\varepsilon)$ such that $\alpha = \overline{\partial}\beta$ on the polydisc.*

Proof. See [Huy05, Prop. 1.3.8]. □

We can now prove the Dolbeault–Grothendieck lemma:

Proposition 2.38. *Let D_ε be a polydisc in $\mathbb{C}^n$. Then*

$$H^{p,q}_{\overline{\partial}}(B_\varepsilon) = \begin{cases} \textit{holomorphic (p-forms)} & q = 0 \ , \\ 0 & q > 0 \ . \end{cases}$$

Proof. The idea is to exhaust the polydisc D_ε by a sequence of approximating polydiscs D_{ε_i}, and show that we can choose the approximating exact terms so that they do not change inside the smaller polydisc.
If $q > 1$, the difference $\beta_i - \beta_{i-1}$ will then be $\overline{\partial}$-closed, so by the $\overline{\partial}$-Poincaré lemma, we can choose γ_i such that $\overline{\partial}\gamma = \beta_i - \beta_{i-1}$.
Take ψ a bump function supported on D_{ε_i} with $\psi|_{D_{\varepsilon_i}} = 1$ and set $\hat{\beta}_{i+1} = \beta_{i+1} + \overline{\partial}(\psi\gamma)$. The sequence $\hat{\beta}_i$ has the desired properties.
The case $q = 1$ follows a similar idea, where now $\overline{\partial}\gamma$ is replaced by a suitable holomorphic polynomial.
The full details can be found in [Huy05, Cor. 1.3.9]. □

3 Sheaves and their cohomologies

We now introduce the language and techniques from sheaf theory. While we will not use them to their fullest extent, they are a convenient tool for presenting and proving some of our results, especially when considering cohomology and vector bundles. A more detailed discussion can be found in [Wel08] and references therein. For a more detailed and comprehensive discussion using derived functors, we refer the reader to [Har77, §3].

Definition 3.1. A *presheaf* $\mathcal{F}$ of abelian groups (sets, rings, ...) on a topological space X is given by:

(i) For every open set $U \subseteq X$, an abelian group $\mathcal{F}(U)$

(ii) For every inclusion $V \subseteq U$, a mapping $\mathcal{F}(U) \to \mathcal{F}(V)$ (restriction map)

such that $r_{UU} = \mathrm{id}$ and $r_{VW} \circ r_{UV} = r_{UW}$ for $W \subseteq V \subseteq U$.

Definition 3.2. A presheaf is called a *sheaf* if for every family of sections $s_i \in \mathcal{F}(U_i)$, $i \in I$, with $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$. Equivalently, the sequence:

$$0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \to \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is exact, where the map is $(s_i) \mapsto (s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j})$.

We can now give a (perhaps) more intuitive definition of a stalk as a direct limit of a presheaf.

Definition 3.3. The *stalk* of a presheaf $\mathcal{F}$ at $x \in X$ is:

$$\mathcal{F}_x := \varinjlim_{x \in U} \mathcal{F}(U) = \bigcup_{x \in U} \mathcal{F}(U)/\sim$$

where $s_U \sim s_V$ if $s_U|_W = s_V|_W$ for some $x \in W \subseteq U \cap V$.

Associated with a presheaf, we have an associated topological space:

Definition 3.4. For a presheaf $\mathcal{F}$, define:

$$\acute{\mathrm{E}}\mathrm{t}(\mathcal{F}) := \bigcup_{x \in X} \mathcal{F}_x \xrightarrow{p} X \quad \text{with} \quad p^{-1}(x) = \mathcal{F}_x$$

The sets $[U, s] = \{s_x \mid x \in U\}$ for U open, $s \in \mathcal{F}(U)$, form a basis for a topology on $\acute{\mathrm{E}}\mathrm{t}(\mathcal{F})$, and p is a local homeomorphism.
The *sheafification*$\mathcal{F}^+$ of a presheaf $\mathcal{F}$ is defined by:

$$\mathcal{F}^+(U) = \{s : U \to \acute{\mathrm{E}}\mathrm{t}(\mathcal{F}) \mid s \text{ is a continuous section}\}$$

There is a natural map $\mathcal{F}(U) \to \mathcal{F}^+(U)$ compatible with restrictions. If $\mathcal{F}$ is a sheaf, this map is an isomorphism.

An easy (but important) example is that of the constant presheaf and the locally constant sheaf:

Example 3.5. *If $\mathcal{F}^{const}$ is the constant presheaf with $\mathcal{F}^{const}(U) = A$, then:*

$$\acute{E}t(\mathcal{F}^{const}) = X \times A^{disc}, \qquad (\mathcal{F}^{const})^+ = \underline{A}$$

Given a morphism of sheaves, we can study the associated kernel and image. First, we have

Lemma 3.6. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then the presheaf $\ker\varphi$ is a sheaf.*

Proof. To prove that $\ker\varphi$ is a sheaf, we need to prove that, for Uopen and $\{U_i\}$ an open cover of U, we have

(i) (Existence) if $s_i \in \ker\varphi(U_i)$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$, then there exists $s \in \ker\varphi(U)$ such that $s|_{U_i} = s_i$ for all i;

(ii) (Uniqueness) if $s \in \ker\varphi(U)$ and $s|_{U_i} = 0$, then $s = 0$.

To show (i), notice that the candidate s exists in $\mathcal{F}(U)$ since $\mathcal{F}$ is a sheaf. Thus, we only need to show that $s \in \ker\varphi(U)$.
Indeed $\varphi(s_i) = 0$ by hypothesis, and since $\mathcal{G}$ is also a sheaf, this glue together to show that $\varphi(s) = 0$, as needed.
Uniqueness follows readily since $\mathcal{F}$ is a sheaf. □

In general, however the presheaves $U \mapsto \operatorname{im}\varphi_U$ and $U \mapsto \operatorname{coker}\varphi_U$ are not sheaves. For instance, one may consider the image presheaf of the exponential map $\exp : \mathcal{O}_{\mathbb{C}} \to \mathcal{O}^*_{\mathbb{C}}$. Then, for an open set U, $\exp(U)$ is the ring of holomorphic functions on U with a well-defined logarithm. But taking $U_1 = \mathbb{C} \setminus \{x \geq 0\}$ and $U_2 = \mathbb{C} \setminus \{x \leq 0\}$ suffices to see that the image presheaf is not a sheaf, as there is no logarithm defined in $\mathbb{C} \setminus \{0\}$.

Definition 3.7. For a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves, we define:

- The *image sheaf*: $\operatorname{im}\varphi := (U \mapsto \operatorname{im}\varphi_U)^+$
- The *cokernel sheaf*: $\operatorname{coker}\varphi := (U \mapsto \operatorname{coker}\varphi_U)^+$

A sequence $\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$ is called *exact* at $\mathcal{G}$ if $\ker\psi = \operatorname{im}\varphi$.
Similarly, we say the morphism φ is *injective* if $0 \to \mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ is exact; and *surjective* if $\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \to 0$ is exact.

We have the following useful characterisation of exactness:

Lemma 3.8. *The sequence $\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$ is exact iff $\mathcal{F}_x \xrightarrow{\varphi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x$ is exact for all $x \in X$.*

Proof. Exercise. □

The following sequences are examples of exact sequences:

$$0 \to \underline{\mathbb{Z}} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$$
$$0 \to \mathcal{O}_X \to \mathcal{O}_X \to S(0) \to 0$$
$$0 \to \underline{\mathbb{C}} \to \mathcal{A}_X^0 \xrightarrow{d} \mathcal{A}_X^1 \to \dots$$
$$0 \to \Omega_X^p \to \mathcal{A}_X^{p,0} \xrightarrow{\overline{\partial}} \mathcal{A}_X^{p,1} \to \cdots$$

Given a continuous map $f : X \to Y$ between topological spaces, we get induced maps on sheaves on them.

Definition 3.9. Let $f : X \to Y$ a continuous map, $\mathcal{F}$ a sheaf on X and $\mathcal{G}$ a sheaf on Y.

- The *direct image sheaf* of $\mathcal{F}$ is defined as $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$ for $U \subseteq Y$.
- The *inverse image sheaf* of $\mathcal{G}$ is defined as $f^{-1}\mathcal{G}(U) = \lim\limits_{f(U)\subseteq V} \mathcal{G}(V)$, where the direct limit runs over all open subsets V of Y that contain $f(U)$.

One needs to check that the definitions are indeed well-posed, i.e. that the presheaves that are defined are indeed sheaves, but we omit that.
The direct and inverse image sheaves satisfy some nice properties:

Lemma 3.10. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps. Then,*

- $g_* \circ f_* = (g \circ f)_*$, $f^{-1} \circ g^{-1} = (g \circ f)^{-1}$
- f^{-1} *is exact (i.e. it preserves exactness)*
- f_* *and* f^{-1} *are adjoint to each other:* $\mathrm{Hom}(f^{-1}\mathcal{F}, \mathcal{G}) = \mathrm{Hom}(\mathcal{F}, f_*\mathcal{G})$.

Lemma 3.11. *Consider $\iota Z \hookrightarrow X$ a continuous embedding, and $\mathcal{F}$ a sheaf on X. Let $\mathcal{F}|_Z = \iota^{-1}\mathcal{F}$. Then,*

- *if $Z = \{x\}$ is a point, $\mathcal{F}|_Z = \mathcal{F}_x$,*
- *if Z is closed, $\mathcal{F}(Z) = \mathcal{F}|_Z(Z)$, and*
- *if Z is open, $\mathcal{F}|_Z(V) = \mathcal{F}(Z \cap V)$.*

We omit the proofs of these lemmas. Finally, for completeness, we introduce the following definitions

Definition 3.12. A *ringed space* is a pair $(X, \mathcal{R})$ where $\mathcal{R}$ is a sheaf of rings on X.
A *morphism* of ringed spaces $(X, \mathcal{R}) \to (Y, \mathcal{S})$ is a continuous map $f : X \to Y$ together with a morphism of sheaves of rings $f^{-1}\mathcal{S} \to \mathcal{R}$.

Definition 3.13. Let $(X, \mathcal{R})$ be a ringed space. A *sheaf of $\mathcal{R}$-modules* is a sheaf of abelian groups $\mathcal{M}$ with a map $\mathcal{R} \times \mathcal{M} \to \mathcal{M}$ such that $\mathcal{M}(U)$ is an $\mathcal{R}(U)$-module for all open U.

Examples of ringed spaces are smooth manifolds: $(X, \mathcal{C}^\infty_X)$, and complex manifolds: $(X, \mathcal{O}_X)$. Examples of $\mathcal{R}$-modules are discussed in the exercises.

3.1 Sheaf cohomology

Let us now discuss the issue of exactness (or rather its failure). We saw (or rather left as an exercise) that taking stalks is an exact operation. More generally, we have

Lemma 3.14. *Let*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of sheaves. Then, for any U, we have

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$$

Proof. □

In general, we lose exactness on the right, as exemplified by the fact that the exponential map $\exp : \mathcal{O}_\mathbb{C} \to \mathcal{O}^*_\mathbb{C}$ is not surjective when evaluated over $U = \mathbb{C} \setminus \{0\}$.
Cohomology is then introduced as a measure of failure for right-exactness. The correct way to understand sheaf cohomology is via the theory of derived functors, which is unfortunately beyond the scope of this course. Instead, we will present an ad-hoc construction for it.

Definition 3.15. A sheaf $\mathcal{I}$ is *injective* if for any injection $\mathcal{A} \hookrightarrow \mathcal{B}$ and map $\mathcal{A} \to \mathcal{I}$, there exists a map $\mathcal{B} \to \mathcal{I}$ making the diagram commute.

Definition 3.16. A *complex of sheaves* is a sequence:

$$\cdots \to \mathcal{F}^{i-1} \xrightarrow{d} \mathcal{F}^i \xrightarrow{d} \mathcal{F}^{i+1} \to \cdots$$

A *resolution* of a sheaf $\mathcal{F}$ is a complex $\mathcal{F}^\bullet$ with a map $\mathcal{F} \hookrightarrow \mathcal{F}^0$ that is exact.
An *injective resolution* is a resolution where all $\mathcal{I}^i$ are injective.

Definition 3.17. The *sheaf cohomology* is defined as:

$$H^i(X, \mathcal{F}) := H^i(\Gamma(X, \mathcal{I}^\bullet))$$

for an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$.

Notice that, in particular $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F}) = \mathcal{F}(X)$. A priori, this definition is subject to the existence of injective resolutions and a choice thereof. Fortunately, we have:

Proposition 3.18.

(i) *Every sheaf $\mathcal{F}$ admits an injective resolution. (The category of sheaves has enough injectives.)*

(ii) *For a morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ and injective resolutions $\mathcal{I}^\bullet$ and $\mathcal{J}^\bullet$ of $\mathcal{F}$ and $\mathcal{G}$, there exist $\varphi^k : \mathcal{I}^k \to \mathcal{J}^k$ such that [TODO] commutes. Moreover, any choice of maps $\{\varphi^k\}$ induces the same maps on cohomology.*

(iii) *Injective sheaves are flabby, i.e. the map $\mathcal{F}(U) \to \mathcal{F}(V)$ for any $V \subseteq U$ open.*

(iv) *If*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is exact and $\mathcal{F}$ is flabby, then

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U) \to 0$$

for all open subsets U.

In particular, this implies that the sheaf cohomology groups are well-defined, and we have

Theorem 3.19. *Consider the short exact sequence of sheaves*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0 \ .$$

Then there exists a long exact sequence of cohomology:

$$0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^0(X, \mathcal{H}) \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G}) \to H^1(X, \mathcal{H}) \to H^2(X, \mathcal{F}) \to \dots$$

is exact

Proof. Use the fact that the injective resolution is flabby, along with the snake lemma/ diagram chasing, to construct the connecting morphisms. □

Whilst injective sheaves and injective resolutions are convenient to define sheaf cohomology, they tend to be quite cumbersome and hard to construct in explicit situations. Instead, it is more convenient to work with acyclic sheaves and resolutions

Definition 3.20. A sheaf $\mathcal{A}$ is *acyclic* if $H^i(X, \mathcal{A}) = 0$ for $i > 0$. An *acyclic resolution* is a resolution $\mathcal{A}^\bullet$ by acyclic sheaves $\mathcal{A}^i$.

The following result captures the convenience of working with acyclic resolution.

Theorem 3.21. *Let $\mathcal{A}^\bullet$ be an acyclic resolution of $\mathcal{F}$, then:*

$$H^i(X, \mathcal{F}) = H^i(\Gamma(X, \mathcal{A}^\bullet))$$

Proof. Split the resolution into short exact sequences:

$$0 \to \mathcal{K}^i \to \mathcal{A}^i \to \mathcal{K}^{i+1} \to 0$$

with $\mathcal{K}^i := \ker\left(A^i \to A^{i+1}\right) \cong \operatorname{im}\left(A^{i-1} \to A^i\right)$.. The long exact sequence of cohomology yields the desired result. □

Exercise 3.22. *Write down the missing details of the proof above.*

We now claim a fact that will be of great importance, but we do not have the time to prove it:

Theorem 3.23. *All sheaves of $\mathcal{A}_{\mathbb{R}}$-modules are acyclic.*

The proof of the theorem relies on constructing a particular type of acyclic sheaves called *soft*, via a partition of unity on X. This dependence on the partition of unity is key in the construction.
As a corollary of this fact, we have

Corollary 3.24. *Let X be a smooth manifold. Then*

$$H^k_{dR}(X, \mathbb{R}) \cong H^k(X, \underline{\mathbb{R}}) .$$

Similarly, on a complex manifold, we have

$$H^{p,q}_{\overline{\partial}}(X) \cong H^q(X, \Omega^p)$$

Proof. The smooth Poincaré lemma implies that the locally constant sheaf $\underline{\mathbb{R}}$ admits the acyclic resolution

$$\mathcal{A}^{\bullet}_{X,\mathbb{R}} := \quad 0 \to \mathcal{A}^0_{X,\mathbb{R}} \xrightarrow{d} \mathcal{A}^1_{X,\mathbb{R}} \xrightarrow{d} \mathcal{A}^2_{X,\mathbb{R}} \xrightarrow{d} \dots$$

Similarly, the $\overline{\partial}$-Poincaré lemma implies that sheaf of holomorphic p-forms admits the acyclic resolution

$$\mathcal{A}^{p,\bullet}_{X,\mathbb{R}} := \quad 0 \to \mathcal{A}^{p,0}_{X,\mathbb{R}} \xrightarrow{\overline{\partial}} \mathcal{A}^{p,1}_{X,\mathbb{R}} \xrightarrow{\overline{\partial}} \mathcal{A}^{p,2}_{X,\mathbb{R}} \xrightarrow{\overline{\partial}} \dots$$

□

3.2 Čech cohomology

We now introduce another, more combinatorial, cohomology theory for sheaves. Whilst it is more "hands-on" and computationally easy to work with, one does not have all the good properties of sheaf cohomology "on the nose".

Definition 3.25. Let $\mathcal{F}$ be a sheaf on X and $\mathcal{U} = \{U_i\}_{i \in I}$ an open cover. For each $\sigma = (i_0, \dots, i_q) \in I^{q+1}$, consider $U_\sigma = U_{i_0} \cap \dots U_{i_q}$ and $\iota_\sigma : U_\sigma \hookrightarrow X$ the inclusion.

(i) The *sheaf of Čech chains* with respect to the cover $\mathcal{U}$ is:

$$\mathcal{C}^q(\mathcal{U},\mathcal{F}) = \prod_{\sigma\in I^{q+1}} (\iota_\sigma)_*(\iota_\sigma)^{-1}\mathcal{F}$$

(ii) The *Čech boundary operator* is:

$$\begin{aligned}\delta : \mathcal{C}^q(\mathcal{U},\mathcal{F}) &\to \mathcal{C}^{q+1}(\mathcal{U},\mathcal{F})\\ (s_\sigma)_\sigma &\mapsto \sum_{k=0}^{q+1}(-1)^k\ (s_{i_0,\ldots,\check{i_k},\ldots i_{q+1}})|_{U_{i_0,\ldots,i_{q+1}}}\end{aligned}$$

A (tedious) computation shows that $\delta^2 = 0$, so $\big(\mathcal{C}^q(\mathcal{U},\mathcal{F}),\delta\big)$ is a complex of sheaves. In particular, we can define the (relative) Čech cohomology groups:

$$\check{H}^q(\mathcal{U},\mathcal{F}) := \frac{\ker\Big(\mathcal{C}^q(\mathcal{U},\mathcal{F}) \xrightarrow{\delta} \mathcal{C}^{q+1}(\mathcal{U},\mathcal{F})\Big)}{\operatorname{im}\Big(\mathcal{C}^{q-1}(\mathcal{U},\mathcal{F}) \xrightarrow{\delta} \mathcal{C}^{q}(\mathcal{U},\mathcal{F})\Big)} .$$

In degree zero, we have

$$C^0(\mathcal{U},\mathcal{F}) = \prod_{U_i}\mathcal{F}(U_i) \xrightarrow{\delta} \prod_{U_i\cap U_j}\mathcal{F}(U_i\cap U_j) = C^1(\mathcal{U},\mathcal{F})$$

with $\delta(s)_{ij} = s_j|_{U_i\cap U_j} - s_i|_{U_i\cap U_j}$. Since $\mathcal{F}$ is a sheaf, $\check{H}^0(\mathcal{U},\mathcal{F}) = \ker\delta = H^0(X,\mathcal{F})$. However, the higher cohomology groups will depend on the chosen cover. To remedy this, we define

Definition 3.26. Let X be a topological space and $\mathcal{F}$ a sheaf. We define the *Čech cohomology groups* as

$$\check{H}^q(X,\mathcal{F}) = \lim_{\mathcal{U}\text{ cover}} \check{H}^q(\mathcal{U},\mathcal{F}) ,$$

where the direct limit is taken over finer and finer covers.

The result that ties up all the discussion is a celebrated result due to Leray:

Theorem 3.27 (Leray's theorem)**.** *There is an isomorphism:*

$$H^q(X,\mathcal{F}) \cong \check{H}^q(X,\mathcal{F})$$

I have been particularly vague and stated many (deep and hard) results at face value, which the reader should be pretty unhappy about (I know I am). Unfortunately, I find it the lesser of all evils, as proceeding in our discussion without the tools of sheaf theory and its cohomologies would prove nearly impossible. However, establishing and discussing all the material summarised in this section in detail could take an entire course on its own.

4 Holomorphic bundles, Kodaira dimension and Siegel's theorem

Recall the definition of smooth real (resp. complex) vector bundles:

Definition 4.1. A real (resp. complex) *vector bundle* of rank r over a manifold X is a smooth manifold E together with a smooth projection $\pi : E \to X$ such that:

- For each $x \in X$, the fibre $E_x = \pi^{-1}(x)$ is a real (resp. complex) vector space of dimension r.
- There exists an open cover $\{U_i\}$ of X and diffeomorphisms $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^r$ (resp. $\mathbb{C}^r$) such that:
  1. $\pi = \mathrm{pr}_1 \circ \varphi_i$ on $\pi^{-1}(U_i)$, where pr_1 denotes the projection to the first factor.
  2. On $U_i \cap U_j$, the transition functions $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : (U_i \cap U_j) \times \mathbb{R}^r \to (U_i \cap U_j) \times \mathbb{R}^r$ are of the form $(x, v) \mapsto (x, g_{ij}(x)v)$ where $g_{ij} \in \mathbb{C}^\infty\big(U_i \cap U_j, \mathrm{GL}(r, \mathbb{R})\big)$

Therefore, one makes the analogue definition for the holomorphic case:

Definition 4.2 (Holomorphic Vector Bundle)**.** A *holomorphic vector bundle* of rank r on a complex manifold X is a complex manifold E together with a holomorphic projection $\pi : E \to X$ such that:

- For each $x \in X$, the fibre $E_x = \pi^{-1}(x)$ is a complex vector space of dimension r.
- There exists an open cover $\{U_i\}$ of X and biholomorphic maps $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^r$ such that:
  1. $\pi = \mathrm{pr}_1 \circ \varphi_i$ on $\pi^{-1}(U_i)$
  2. On $U_i \cap U_j$, the transition functions $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : (U_i \cap U_j) \times \mathbb{C}^r \to (U_i \cap U_j) \times \mathbb{C}^r$ are of the form $(x, v) \mapsto (x, g_{ij}(x)v)$ where $g_{ij} : U_i \cap U_j \to GL_r(\mathbb{C})$ are holomorphic.

Vector bundles are classified by the appropriate (Čech) cohomology group:

Proposition 4.3. *Up to isomorphism, we have the following correspondences:*

- *real vector bundles of rank* $r \xleftrightarrow{1:1} \check{H}^1\big(X, \mathrm{GL}(r, \mathcal{C}^\infty(X, \mathbb{R}))\big)$,
- *complex vector bundles of rank* $r \xleftrightarrow{1:1} \check{H}^1\big(X, \mathrm{GL}(r, \mathcal{C}^\infty(X, \mathbb{C}))\big)$
- *holomorphic vector bundles of rank* $r \xleftrightarrow{1:1} \check{H}^1\big(X, \mathrm{GL}(r, \mathcal{O}_X)\big)$,

where $\mathrm{GL}(r, \mathcal{F})$ *is the sheaf of invertible rank* k *matrices with coefficients in the sheaf* $\mathcal{F}$*.*

Proof. Exercise. □

Understanding the groups $\check{H}^1\big(X, \mathrm{GL}(r, \mathcal{A})\big)$ in general can be very hard, and there are no general results, except for the case $r = 1$:

Lemma 4.4. *Complex line bundles over X are in one-to-one correspondence with elements of $H^2(X, \mathbb{Z})$. Similarly, real line bundles over X are in one-to-one correspondence with elements of $H^1(X, \mathbb{Z}/2\mathbb{Z})$.*

Proof. Consider the exponential sequence

$$0 \to \underline{\mathbb{Z}} \xrightarrow{2\pi i} \mathcal{A}_{\mathbb{C}} \xrightarrow{\exp} \mathcal{A}^*_{\mathbb{C}} \to 0 \ .$$

We have a long exact sequence of cohomology

$$\cdots \to H^1(X, \mathcal{A}_{\mathbb{C}}) \to H^1(X, \mathcal{A}^*_{\mathbb{C}}) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{A}_{\mathbb{C}}) \to \dots \ .$$

Since $\mathcal{A}_{\mathbb{C}}$ is acyclic, the map $c_1 : H^1(X, \mathcal{A}^*_{\mathbb{C}}) \to H^2(X, \mathbb{Z})$ is a bijection.
Similarly, for the real line bundle case, consider the short exact sequence

$$0 \to \mathcal{A}_{\mathbb{R}} \to \mathcal{A}^*_{\mathbb{R}} \to \underline{\mathbb{Z}/2\mathbb{Z}} \to 0 \ .$$ □

In fact, whilst $H^1(X, \mathrm{GL}(r, \mathcal{F})$ does not carry any additional structure, $H^1(X, \mathrm{GL}(1, \mathcal{F}) \cong H^1(X, \mathcal{F}^*)$ always carries the additional structure of an abelian group:

Lemma 4.5. *The set $H^1(X, \mathcal{F}^*)$ carries the structure of an abelian group, where the tensor product induces the group operation, and inverses are given by dualisation*

$$L^{-1} := L^* \cong \mathrm{Hom}(L, \mathbb{C}) \ .$$

Proof. Immediate. □

Corollary 4.6. *The maps*

$$c_1 : H^1(X, \mathcal{A}^*_{\mathbb{C}}) \to H^2(X, \mathbb{Z}) \qquad\qquad w_1 : H^1(X, \mathcal{A}^*_{\mathbb{R}}) \to H^1(X, \mathbb{Z}/2\mathbb{Z})$$

are group morphisms.

Let us now focus on the case of holomorphic line bundles:

Definition 4.7. The group of isomorphism classes of line bundles is called the *Picard group*:

$$\mathrm{Pic}(X) = H^1(X, \mathcal{O}^*_X).$$

Again, by using the exponential short exact sequence, we have:

Proposition 4.8. *Every complex line bundle admits a holomorphic structure. The set of (non-isomorphic) holomorphic structures on a smooth line bundle is in bijection with* $H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$

Proof. Comparing the smooth and holomorphic exponential sequences, we have:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \underline{\mathbb{Z}/2\mathbb{Z}} & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_X^* & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \underline{\mathbb{Z}/2\mathbb{Z}} & \longrightarrow & \mathcal{A}_{\mathbb{C}} & \longrightarrow & \mathcal{A}_{\mathbb{C}}^* & \longrightarrow & 0
\end{array}$$

The claim follows from the induced map of long exact sequences. □

Let us now introduce the first example of (non-trivial) holomorphic line bundle; the tautological line bundle of the complex projective space $\mathbb{CP}^n$:

Proposition 4.9. *The tautological line bundle* $\mathcal{O}(-1)$ *on* $\mathbb{P}^n$ *is defined by:*

$$\mathcal{O}(-1) = \{(l, z) \mid z \in l\} \subseteq \mathbb{P}^n \times \mathbb{C}^{n+1}$$

with projection $\pi : \mathcal{O}(-1) \to \mathbb{P}^n$*.*

Proof. On affine charts $U_i = \{z_i \neq 0\}$, we have trivializations:

$$\pi^{-1}(U_i) \cong U_i \times \mathbb{C}, \quad (l, z) \mapsto (l, z_i)$$

The transition functions are $\psi_{ij}(l) = \frac{z_i}{z_j}$. □

Definition 4.10. For $k > 0$, define $\mathcal{O}(k) = \mathcal{O}(-1)^{\otimes -k}$, and $\mathcal{O}(0) = \mathcal{O}_{\mathbb{P}^n}$.

Proposition 4.11. *The global sections of* $\mathcal{O}(k)$ *are given by:*

$$H^0(\mathbb{P}^n, \mathcal{O}(k)) = \mathbb{C}[z_0, \ldots, z_n]_k \quad \text{for } k \geq 0$$

Proof. Any homogeneous polynomial $P \in \mathbb{C}[z_0, \ldots, z_n]_k$ defines a section s_P of $\mathcal{O}(k)$ via:

$$s_P(l) = (l, P(z)) \quad \text{for } z \in l$$

Conversely, any section $s \in H^0(\mathbb{P}^n, \mathcal{O}(k))$ gives a function $F : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}$ satisfying $F(\lambda z) = \lambda^k F(z)$, which is a homogeneous polynomial of degree k. □

Definition 4.12. The *canonical bundle* of a complex manifold X is the bundle of holomorphic top forms $K_X = \bigwedge^{\dim X} \Omega_X^1$.

Definition 4.13. A compact complex manifold with $K_X \cong \mathcal{O}_X$ is called *(weak) Calabi-Yau.*

5 Divisors and blow-ups

6 Metrics and connections

7 Kähler Manifolds

8 Positivity and vanishing

9 The Kodaira embedding theorem

10 Kodaira-Spencer deformation theory

11 The Tian-Todorov theorem

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