From spheres to new examples of isoparametric hypersurfaces in symmetric spaces

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Joint work with Miguel Domínguez Vázquez

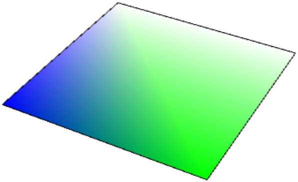
Symmetries in Riemannian Geometry
King's College London

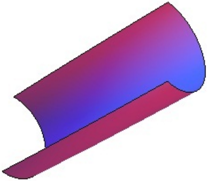
Contents

Introduction

Isoparametric hypersurface

A hypersurface is said to be **isoparametric** $:\Leftrightarrow$ it and its nearby equidistant hypersurfaces have constant mean curvature





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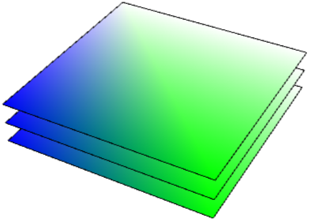

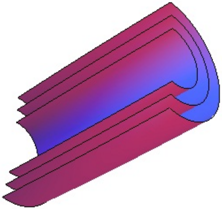

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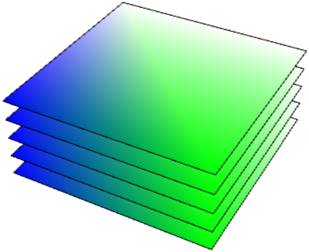

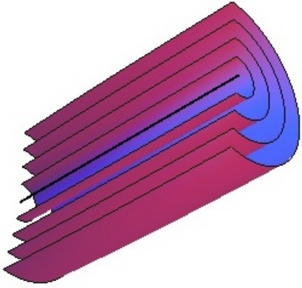

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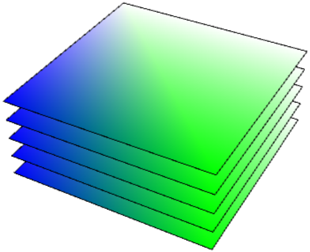

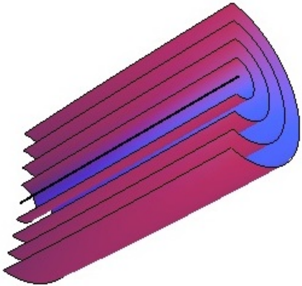

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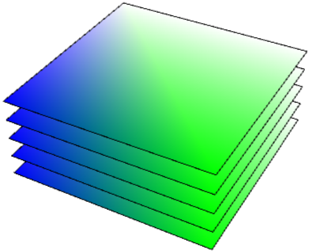

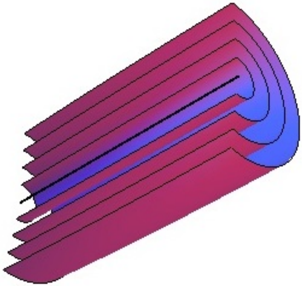

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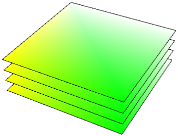
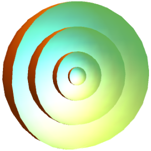
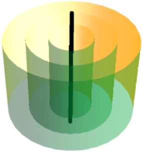

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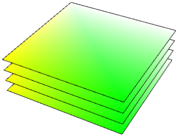

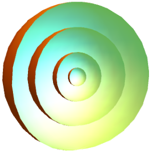

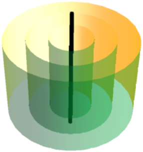

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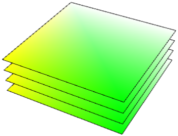
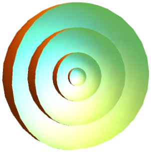
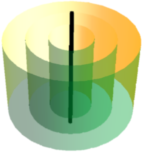

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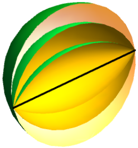
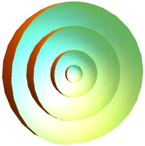
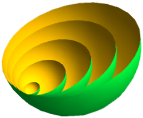

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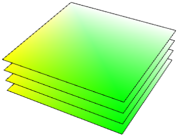
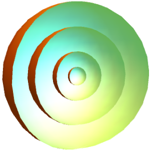
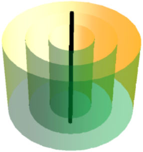

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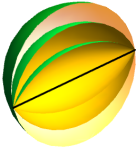
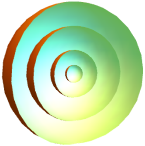
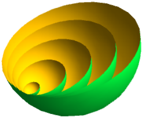

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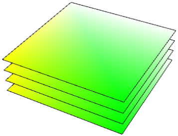
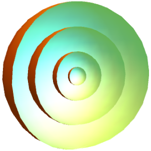
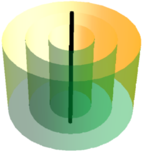

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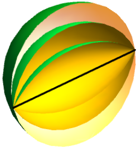
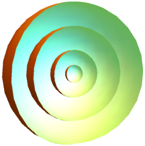
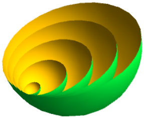

- $\mathbb{S}^n$: Principal curvatures $\Rightarrow$ $\{1, 2, 3, 4, 6\}$
  1. Inhomogeneous isoparametric hypersurfaces (Ferus, Karcher, Münzner)
  2. Isoparametric hypersurfaces are homogeneous or FKM

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Theorem (Ge, Tang, 2012)

1. Focal submanifolds of isoparametric families are minimal
2. Isoparametric + constant principal curvatures $\Rightarrow$ principal curvatures of the focal submanifold independent on the normal

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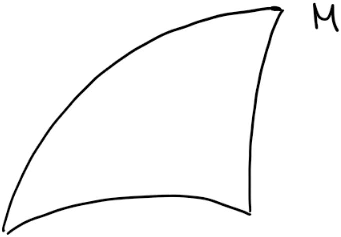

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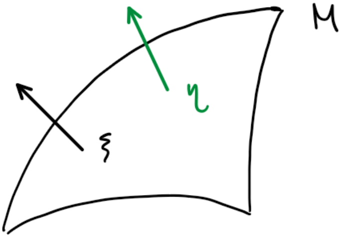

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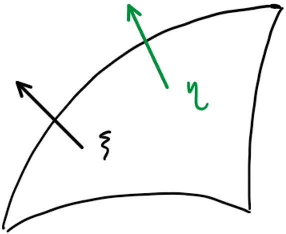

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$:\Leftrightarrow \ \mathrm{eig}\, A_\xi = \mathrm{eig}\, A_\eta$

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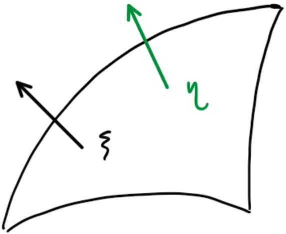

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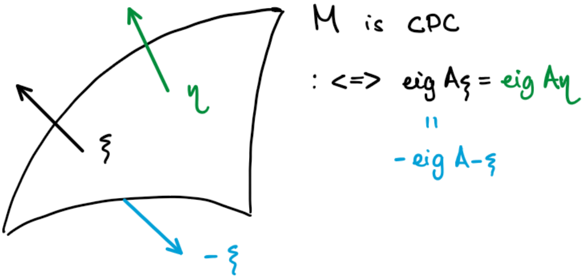

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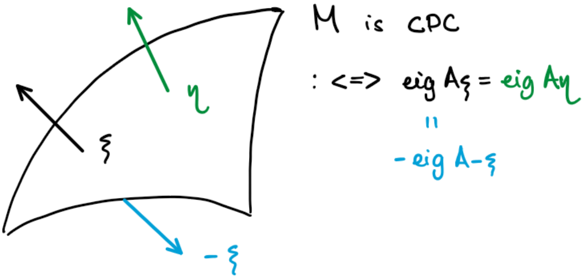
Totally geodesic ⊂ CPC ⊂ Austere ⊂ Minimal

Austere :⇔ Principal curvatures invariant under change of sign



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Totally geodesic $\subset$ CPC $\subset$ Austere $\subset$ Minimal

Austere $:\Leftrightarrow$ Principal curvatures invariant under change of sign

Theorem (Berndt, -, 2018)

- Family of homogeneous non totally geodesic CPC submanifolds

Objectives:

- Understand better cohomogeneity one actions
- Produce new examples of isoparametric hypersurfaces

All the known examples: tubes around an austere focal submanifold

Main Theorem

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- Classification in complete Riemannian families: $\mathbb{R}H^n$, $\mathbb{C}H^n$
- Inhomogeneous examples only in $\mathbb{C}H^n$, $\mathbb{H}H^n$, $\mathbb{O}H^2$ and "extensions"

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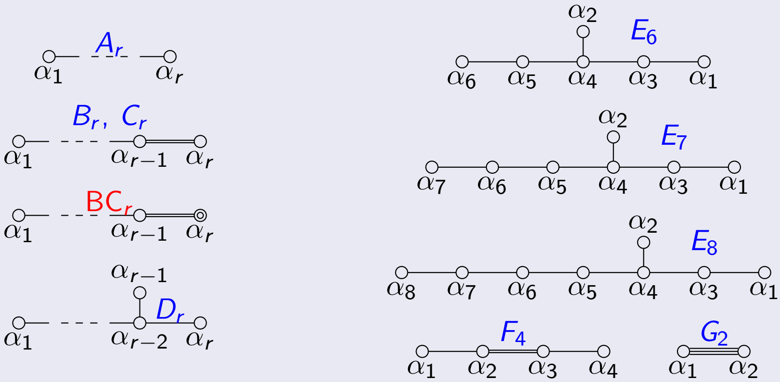
Dynkin diagram



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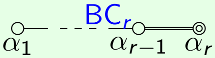

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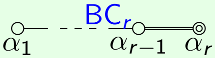

- Inhomogeneous example in $\mathbb{R}H^2 \times \mathbb{R}H^2 \times \mathbb{R}H^2$

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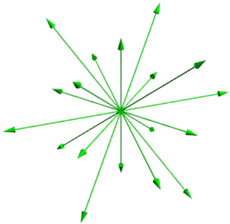
$\lambda(H) = \langle H_\lambda, H \rangle$, for all $H \in \mathfrak{a}$



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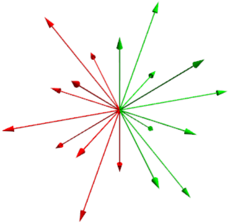
$\lambda(H) = \langle H_\lambda, H \rangle$, for all $H \in \mathfrak{a}$



Symmetric spaces of non-compact type

$M \cong G/K$ symmetric space of non-compact type

- $G = I^0(M)$
- K: isotropy group of $o \in M$

- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition
- Inner product in $\mathfrak{g}$: $\langle X, Y \rangle = -B(\theta X, Y)$, B Killing form
- $\mathfrak{a} \subset \mathfrak{p}$ maximal abelian subspace, $\dim \mathfrak{a} = \operatorname{rank} M$
- $\{\operatorname{ad}(H) : H \in \mathfrak{a}\}$ self-adjoint commutative endomorphisms
- $\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha \right)$ root space decomposition
- $\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : [A, X] = \lambda(A)X, \text{ for all } A \in \mathfrak{a}\}$, $\lambda \in \mathfrak{a}^*$
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Iwasawa decomposition

- $\mathfrak{g} = \mathfrak{k} \oplus \overbrace{\mathfrak{a} \oplus \mathfrak{n}}^{AN}$
- $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$ nilpotent

$M \cong AN$ solvable Lie group with left invariant metric

Construction: focal submanifold

General approach

- $\mathfrak{a} \oplus \mathfrak{n} = \mathfrak{a} \oplus (\bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha)$: Iwasawa decomposition
- $\mathfrak{b} \subset \mathfrak{a} \to \mathfrak{s} = \mathfrak{b} \oplus \mathfrak{n}$ Lie subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$
- S connected Lie subgroup of AN with Lie algebra $\mathfrak{s}$

$$S \cdot o$$

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Ge, Tang: Focal sets of isoparametric families are minimal

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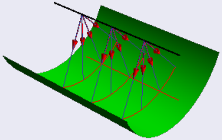

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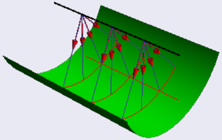

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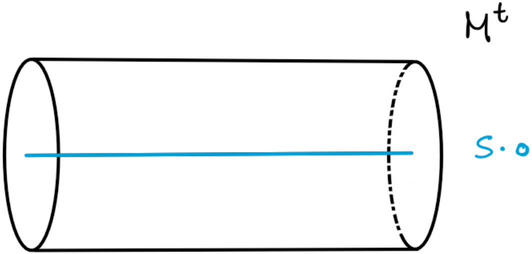

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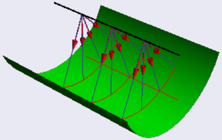

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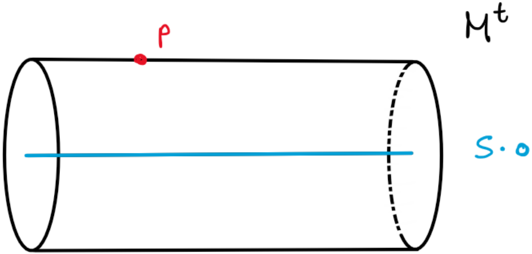

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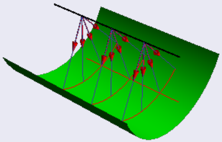

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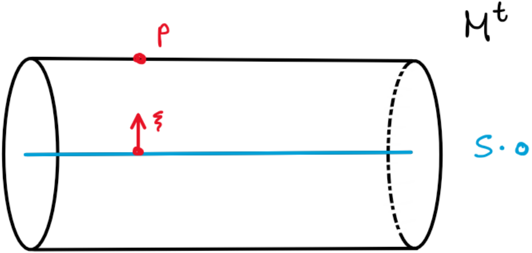

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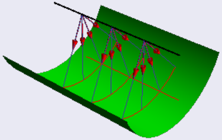

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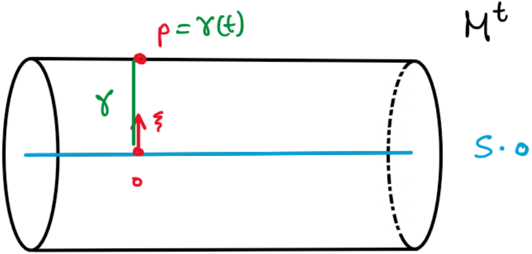

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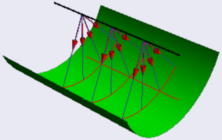

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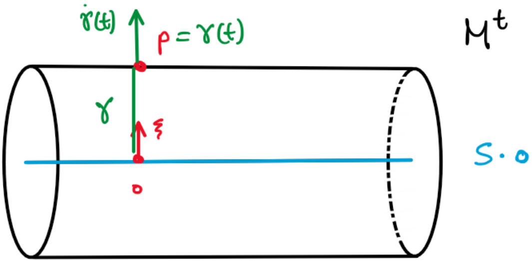

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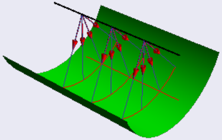

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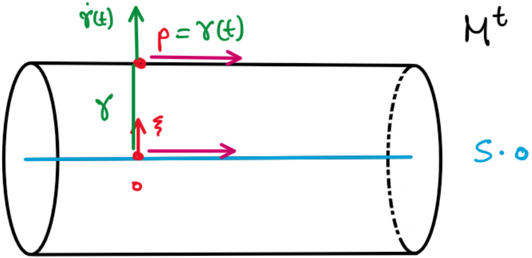

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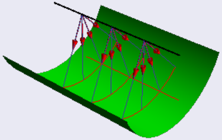

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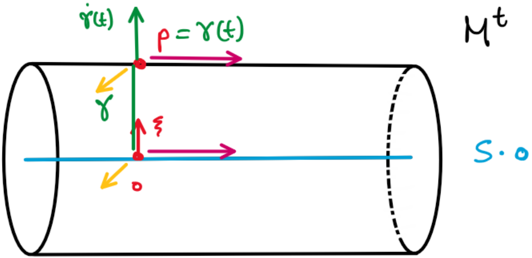

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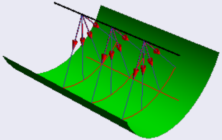

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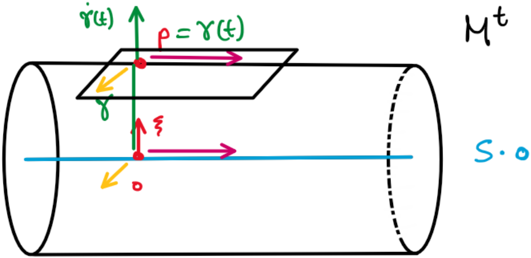

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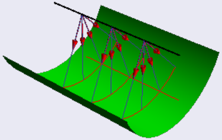

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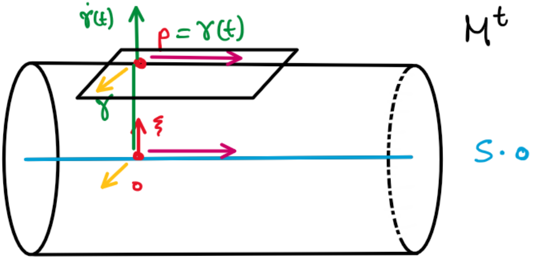

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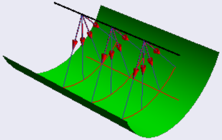

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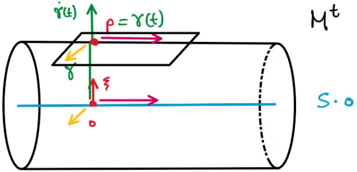

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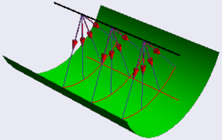

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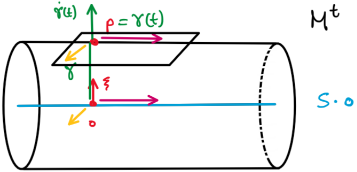

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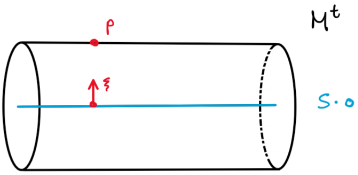

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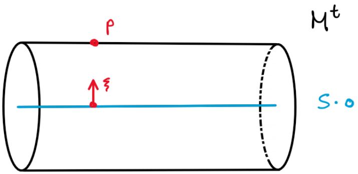

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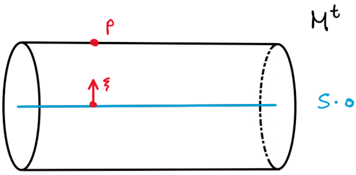

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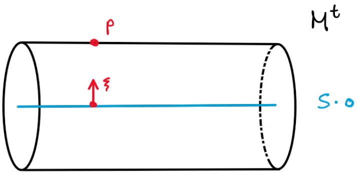

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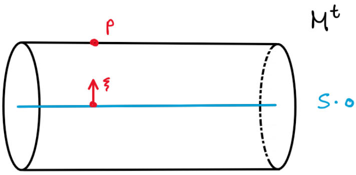

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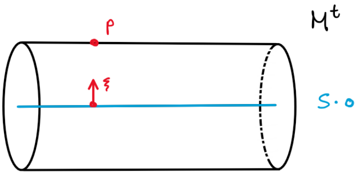

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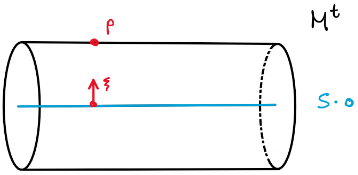

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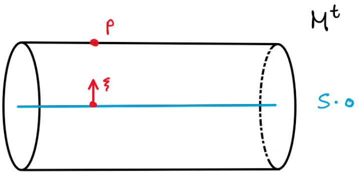

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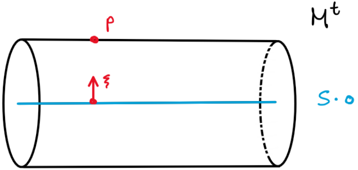

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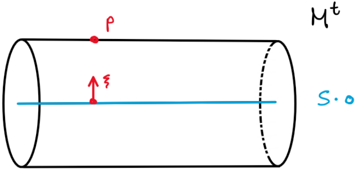

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Non-austere focal submanifold

About the examples

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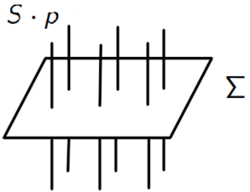
Description of the examples

Polar action

An isometric action on M is said to be polar if there is a submanifold Σ (section) that intersects all orbits orthogonally



Theorem (Domínguez-Vázquez, 2015)

M Riemannian manifold, $S \subset I^0(M)$, $S \circlearrowright M$

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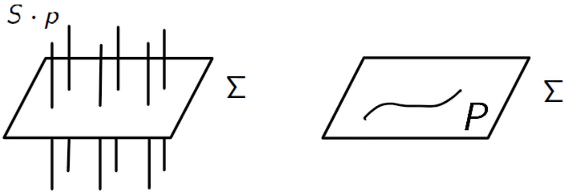

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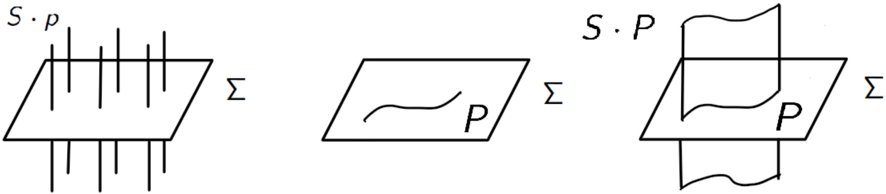

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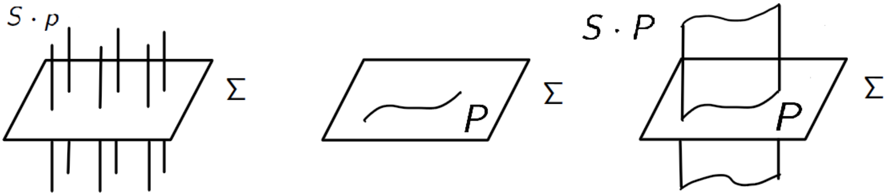

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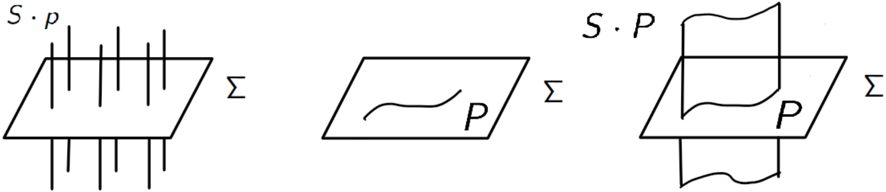

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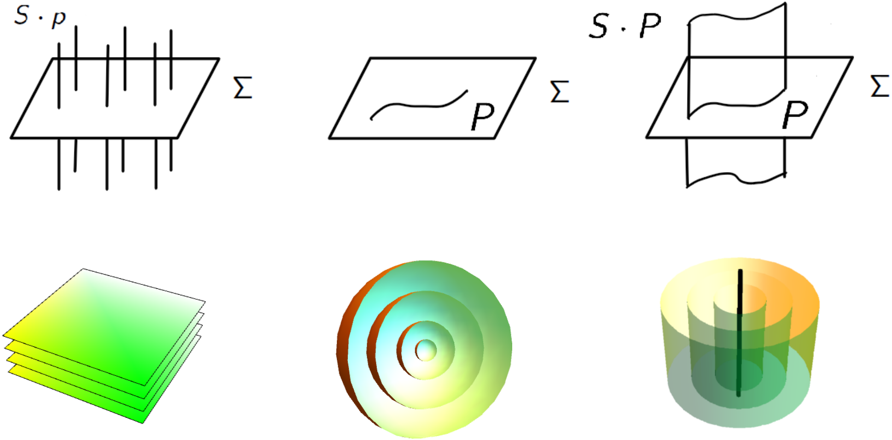

Description of the examples

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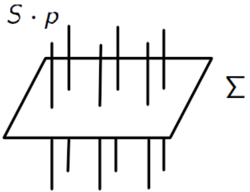

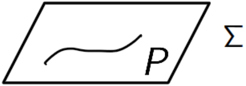

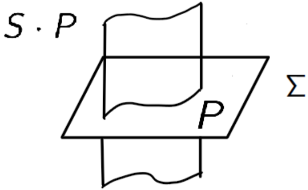

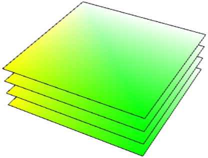

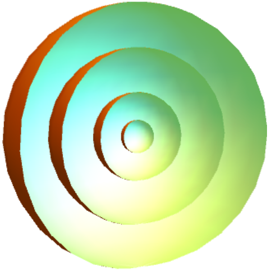

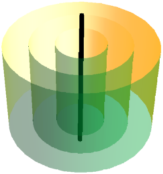

Main Theorem

Theorem

Let M be a symmetric space of non-compact type and rank ≥ 3. Let S be the connected Lie subgroup of AN with Lie algebra $\mathfrak{s} = \mathfrak{b} \oplus \mathfrak{n}$, where $\mathfrak{b}$ is any subspace of codimension at least two of $\mathfrak{a}$ such that $\mathcal{H} \in \mathfrak{b}$. Then:

- The orbit $S \cdot o$ is a minimal submanifold. It is non-austere for a generic choice of $\mathfrak{b}$ as above, or if $\dim \mathfrak{b} = 1$
- The distance tubes around $S \cdot o$ define an inhomogeneous isoparametric family of hypersurfaces with non-constant principal curvatures on M
- There are infinitely many non-congruent examples

M. Domínguez-Vázquez, V. Sanmartín-López: Isoparametric hypersurfaces in symmetric spaces of non-compact type and higher rank. *Compos. Math.* **160** (2024), no. 2, 451–462.