Instantons on ALC G2 metrics

Jakob R. Stein (UniCamp)

Symmetries in Geometry Day

June 10, 2024

Overveiw

- Solution space to (time-dependant) ODE system: **Yang-Mills instanton** equation on co-homogeniety one G_2-manifold, gauge group $SU(2)$.

Overveiw

- Solution space to (time-dependant) ODE system: **Yang-Mills instanton** equation on co-homogeniety one G_2-manifold, gauge group $SU(2)$.
- $\mathbb{B}_7$-family of G_2-metrics on $S^3 \times \mathbb{R}^4$.

Overveiw

- Solution space to (time-dependant) ODE system: **Yang-Mills instanton** equation on co-homogeniety one G_2-manifold, gauge group $SU(2)$.
- $\mathbb{B}_7$-family of G_2-metrics on $S^3 \times \mathbb{R}^4$.
- **Warm-up**: Yang-Mills instantons on **Taub-NUT** $\mathbb{R}^4$.

Overveiw

- Solution space to (time-dependant) ODE system: **Yang-Mills instanton** equation on co-homogeniety one G_2-manifold, gauge group $SU(2)$.
- $\mathbb{B}_7$-family of G_2-metrics on $S^3 \times \mathbb{R}^4$.
- **Warm-up**: Yang-Mills instantons on **Taub-NUT** $\mathbb{R}^4$.
- **Asymptotically locally conical** (ALC) geometry, **asymptotically conical** (AC) limits.

Overveiw

- Solution space to (time-dependant) ODE system: **Yang-Mills instanton** equation on co-homogeniety one G_2-manifold, gauge group $SU(2)$.
- $\mathbb{B}_7$-family of G_2-metrics on $S^3 \times \mathbb{R}^4$.
- **Warm-up**: Yang-Mills instantons on **Taub-NUT** $\mathbb{R}^4$.
- **Asymptotically locally conical** (ALC) geometry, **asymptotically conical** (AC) limits.
- ODE Analysis.

Overveiw

- Solution space to (time-dependant) ODE system: **Yang-Mills instanton** equation on co-homogeniety one G_2-manifold, gauge group $SU(2)$.
- $\mathbb{B}_7$-family of G_2-metrics on $S^3 \times \mathbb{R}^4$.
- **Warm-up**: Yang-Mills instantons on **Taub-NUT** $\mathbb{R}^4$.
- **Asymptotically locally conical** (ALC) geometry, **asymptotically conical** (AC) limits.
- ODE Analysis.
- **ALC/ALF fibrations**.

Motivation

- Partial results by Lotay-Oliveira

Motivation

- Partial results by Lotay-Oliveira
- Recent classification for instantons on multi-Taub-NUT by Cherkis et. al.

Motivation

- Partial results by Lotay-Oliveira
- Recent classification for instantons on multi-Taub-NUT by Cherkis et. al.
- This project joint with Matt Turner,

Invariant Instantons on Taub-NUT

- **Taub-NUT**: for $\eta \in (m^{-2}, \infty)$, e^i dual to vector fields E_i of $SO(3)$ on S^3:

$$g = \eta \left(\eta - \tfrac{1}{m^2}\right)^{-4} d\eta^2 + \eta \left(\eta - \tfrac{1}{m^2}\right)^{-2} \left[\left(e^1\right)^2 + \left(e^2\right)^2\right] + \eta^{-1} \left(e^3\right)^2$$

defines a hyperKahler metric on $\mathbb{R}^4$ asymptotic to a circle fibration over $\mathbb{R}^3$ with a fibres of length m (**ALF**).

Invariant Instantons on Taub-NUT

- **Taub-NUT**: for $\eta \in (m^{-2}, \infty)$, e^i dual to vector fields E_i of $SO(3)$ on S^3:

$$g = \eta \left(\eta - \tfrac{1}{m^2}\right)^{-4} d\eta^2 + \eta \left(\eta - \tfrac{1}{m^2}\right)^{-2} \left[\left(e^1\right)^2 + \left(e^2\right)^2 \right] + \eta^{-1} \left(e^3\right)^2$$

  defines a hyperKahler metric on $\mathbb{R}^4$ asymptotic to a circle fibration over $\mathbb{R}^3$ with a fibres of length m (**ALF**).

- **Yang-Mills Instantons**: for parameters $C, D \geq 0$, $SU(2)$-invariant solutions of $F_A = - * F_A$:

$$\begin{aligned} A = {} & \tfrac{C}{\eta - m^{-2}} \operatorname{csch}\left(\tfrac{C}{\eta - m^{-2}} + D \right) \left[E_1 e^1 + E_2 e^2 \right] \\ & + \tfrac{1}{\eta} \left(m^{-2} + C \coth\left(\tfrac{C}{\eta - m^{-2}} + D \right) \right) E_3 e^3 \end{aligned}$$

Invariant Instantons on Taub-NUT

$$A = \frac{C}{\eta - m^{-2}} \operatorname{csch}\left(\frac{C}{\eta - m^{-2}} + D\right)\left[E_1 e^1 + E_2 e^2\right]$$
$$+ \frac{1}{\eta}\left(m^{-2} + C \coth\left(\frac{C}{\eta - m^{-2}} + D\right)\right) E_3 e^3$$

- Asymptotic to **abelian solution** $\frac{1}{\eta}\left(m^{-2} + C\right) E_3 e^3$: Dirac monopole on $\mathbb{R}^3$. Also (rigid) solutions for $C < 0$.

Invariant Instantons on Taub-NUT

$$A = \frac{C}{\eta - m^{-2}} \operatorname{csch}\left(\frac{C}{\eta - m^{-2}} + D\right)\left[E_1 e^1 + E_2 e^2\right] + \frac{1}{\eta}\left(m^{-2} + C \coth\left(\frac{C}{\eta - m^{-2}} + D\right)\right) E_3 e^3$$

- Asymptotic to **abelian solution** $\frac{1}{\eta}\left(m^{-2} + C\right) E_3 e^3$: Dirac monopole on $\mathbb{R}^3$. Also (rigid) solutions for $C < 0$.
- Fix $B = \sinh(D)/C$ and let $C \to 0$, $D \to 0$:

$$A = \frac{1}{1 + B(\eta - m^{-2})}\left[E_1 e^1 + E_2 e^2\right] + \frac{1}{\eta}\left(m^{-2} + \frac{(\eta - m^{-2})}{1 + B(\eta - m^{-2})}\right) E_3 e^3$$

Asymptotic to **flat** Maurier-Cartan connection $E_1 e^1 + E_2 e^2 + E_3 e^3$

Invariant Instantons on Taub-NUT

$$A = \frac{C}{\eta - m^{-2}} \operatorname{csch}\left(\frac{C}{\eta - m^{-2}} + D\right)\left[E_1 e^1 + E_2 e^2\right] + \frac{1}{\eta}\left(m^{-2} + C \coth\left(\frac{C}{\eta - m^{-2}} + D\right)\right) E_3 e^3$$

- Asymptotic to **abelian solution** $\frac{1}{\eta}\left(m^{-2} + C\right) E_3 e^3$: Dirac monopole on $\mathbb{R}^3$. Also (rigid) solutions for $C < 0$.
- Fix $B = \sinh(D)/C$ and let $C \to 0$, $D \to 0$:

$$A = \frac{1}{1 + B(\eta - m^{-2})}\left[E_1 e^1 + E_2 e^2\right] + \frac{1}{\eta}\left(m^{-2} + \frac{(\eta - m^{-2})}{1 + B(\eta - m^{-2})}\right) E_3 e^3$$

  Asymptotic to **flat** Maurier-Cartan connection $E_1 e^1 + E_2 e^2 + E_3 e^3$
- Euclidian limit $m \to \infty$. $A \to A_{\mathrm{BPS}}$

$$A_{\mathrm{BPS}} = \frac{1}{1 + B\eta^2}\left[E_1 e^1 + E_2 e^2 + E_3 e^3\right]$$

$\mathbb{B}_7$-family

- Given the data of four functions $(A_1, A_3, B_1, B_3) : \mathbb{R}_{>0} \to \mathbb{R}^4_{>0}$ solving:

$$\dot{A}_1 = \frac{1}{2}\left(\frac{B_1^2 + B_3^2 - A_1^2}{B_1 B_3} - \frac{A_3}{A_1}\right),$$
$$\dot{A}_3 = \frac{1}{2}\left(\frac{A_3^2}{A_1^2} - \frac{A_3^2}{B_1^2}\right),$$
$$\dot{B}_1 = \frac{1}{2}\left(\frac{A_1^2 + B_3^2 - B_1^2}{A_1 B_3} + \frac{A_3}{B_1}\right),$$
$$\dot{B}_3 = \frac{A_1^2 + B_1^2 - B_3^2}{A_1 B_1}.$$

$\mathbb{B}_7$-family

- Given the data of four functions $(A_1, A_3, B_1, B_3) : \mathbb{R}_{>0} \to \mathbb{R}^4_{>0}$ solving:

$$\dot{A}_1 = \frac{1}{2}\left(\frac{B_1^2 + B_3^2 - A_1^2}{B_1 B_3} - \frac{A_3}{A_1}\right),$$
$$\dot{A}_3 = \frac{1}{2}\left(\frac{A_3^2}{A_1^2} - \frac{A_3^2}{B_1^2}\right),$$
$$\dot{B}_1 = \frac{1}{2}\left(\frac{A_1^2 + B_3^2 - B_1^2}{A_1 B_3} + \frac{A_3}{B_1}\right),$$
$$\dot{B}_3 = \frac{A_1^2 + B_1^2 - B_3^2}{A_1 B_1}.$$

- We can construct an $SU(2)^2$-invariant metric with holonomy G_2 on $\mathbb{R}_{>0} \times S^3 \times S^3$, w. basis of left-invariant one-forms $e_i, e_i' \in \mathfrak{su}^*(2)$ by

$$g = dt^2 + A_i^2\left(\sum_{i=1}^{3}(e_i + e_i')^2\right) + B_i^2\left(\sum_{i=1}^{3}(e_i - e_i')^2\right) \qquad (1)$$

where $A_1 = A_2,\ B_1 = B_2$.

$\mathbb{B}_7$-family

- **2-parameter family** (r_0, ℓ) of solutions that are smooth over $S^3 \subset S^3 \times \mathbb{R}^4$ at $t = 0$.

$\mathbb{B}_7$-family

- **2-parameter family** (r_0, ℓ) of solutions that are smooth over $S^3 \subset S^3 \times \mathbb{R}^4$ at $t = 0$.
- r_0 is fixed by overall scale, ℓ controls **asymptotic behaviour** (r_0 fixed).

$\mathbb{B}_7$-family

- **2-parameter family** (r_0, ℓ) of solutions that are smooth over $S^3 \subset S^3 \times \mathbb{R}^4$ at $t = 0$.
- r_0 is fixed by overall scale, ℓ controls **asymptotic behaviour** (r_0 fixed).
- $(A_1, A_3, B_1, B_3) \sim \left(\frac{\sqrt{3}}{3}t, \ell, \frac{\sqrt{3}}{3}t, \frac{2}{3}t\right)$ as $t \to \infty$.

$\mathbb{B}_7$-family

- **2-parameter family** (r_0, ℓ) of solutions that are smooth over $S^3 \subset S^3 \times \mathbb{R}^4$ at $t = 0$.
- r_0 is fixed by overall scale, ℓ controls **asymptotic behaviour** (r_0 fixed).
- $(A_1, A_3, B_1, B_3) \sim \left(\frac{\sqrt{3}}{3}t, \ell, \frac{\sqrt{3}}{3}t, \frac{2}{3}t\right)$ as $t \to \infty$.
- $g \sim dt^2 + t^2 g_5 + \ell^2 (e_3 + e_3')^2$.

$\mathbb{B}_7$-family

- **2-parameter family** (r_0, ℓ) of solutions that are smooth over $S^3 \subset S^3 \times \mathbb{R}^4$ at $t = 0$.
- r_0 is fixed by overall scale, ℓ controls **asymptotic behaviour** (r_0 fixed).
- $(A_1, A_3, B_1, B_3) \sim \left(\frac{\sqrt{3}}{3}t, \ell, \frac{\sqrt{3}}{3}t, \frac{2}{3}t\right)$ as $t \to \infty$.
- $g \sim dt^2 + t^2 g_5 + \ell^2 (e_3 + e_3')^2$.
- $dt^2 + t^2 g_5$ is Calabi-Yau cone metric on **conifold** $\{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \mid \sum_i z_i^2 = 0\}$.

$\mathbb{B}_7$-family

- $g \sim dt^2 + t^2 g_5 + \ell^2 \left(e_3 + e_3'\right)^2.$

$\mathbb{B}_7$-family

- $g \sim dt^2 + t^2 g_5 + \ell^2 (e_3 + e_3')^2$.
- Asymptotically locally conical (**ALC**): tangent cone at infinity is 1d less. Volume of ball $\mathrm{Vol}(B_t) = O(t^6)$.

$\mathbb{B}_7$-family

- $g \sim dt^2 + t^2 g_5 + \ell^2 (e_3 + e_3')^2$.
- Asymptotically locally conical (**ALC**): tangent cone at infinity is 1d less. Volume of ball $\mathrm{Vol}(B_t) = O(t^6)$.
- Rescaling $s_{\lambda^2} : t \mapsto \lambda^2 t$, $s_{\lambda^2}^* g \sim \lambda^4 \left(g_{\mathrm{TN}} \oplus \lambda^{-2} g_{\mathrm{round}}\right)$

G2-instantons

- Given the data of two functions $(F_1, F_3) : \mathbb{R}_{>0} \to \mathbb{R}^2_{>0}$ solving:

$$\dot{F}_1 = \frac{F_1}{A_3}\left(1 - \frac{A_1 A_3}{B_1 B_3} - F_3\right),$$
$$\dot{F}_3 = \frac{A_3}{A_1^2}\left(\left(1 - \frac{A_1^2}{B_1^2}\right) F_3 - F_1^2\right),$$

G2-instantons

- Given the data of two functions $(F_1, F_3) : \mathbb{R}_{>0} \to \mathbb{R}^2_{>0}$ solving:

$$\dot{F}_1 = \frac{F_1}{A_3}\left(1 - \frac{A_1 A_3}{B_1 B_3} - F_3\right),$$

$$\dot{F}_3 = \frac{A_3}{A_1^2}\left(\left(1 - \frac{A_1^2}{B_1^2}\right) F_3 - F_1^2\right),$$

- We can construct an $SU(2)^2$-invariant Yang-Mills (G_2) instanton $D^*_{\mathbb{A}} F_{\mathbb{A}} = 0$ on $\mathbb{R}_{>0} \times S^3 \times S^3$ by

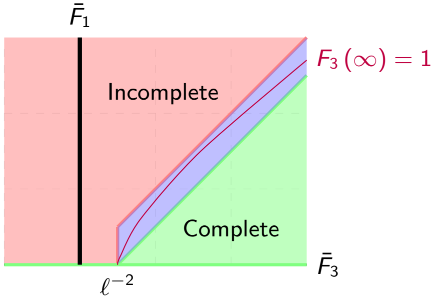
$$\mathbb{A} = \left(\sum_{i=1}^{3} F_i E^i \otimes (e_i + e'_i)\right)$$

where $F_1 = F_2$, $E^i := (e_i)^* \in \mathfrak{su}(2)$.

Space of solutions for fixed ℓ

Solutions closing on S^3 in a two-parameter family:

$$F_1 = \bar{F}_1 t^2 + O(t^4) \qquad\qquad F_3 = \bar{F}_3 t^2 + O(t^4)$$



Theorem

(S.-Turner) The above picture describes the region of initial conditions that lead to complete bounded solutions.

Strategy Of Proof

- Extend **incompleteness** results of Lotay-Oliveria.

Strategy Of Proof

- Extend **incompleteness** results of Lotay-Oliveria.
- Parameterise **solutions at infinity** using theory of **irregular singular initial value problems** of the form:

$$\dot{x} = t^{-2} M_{-2} x + t^{-1} M_{-1} x + M(x, t) \quad [M_{-2}, M_{-1}] = 0$$

Strategy Of Proof

- Extend **incompleteness** results of Lotay-Oliveria.
- Parameterise **solutions at infinity** using theory of **irregular singular initial value problems** of the form:

$$\dot{\mathrm{x}} = t^{-2}\mathrm{M}_{-2}\mathrm{x} + t^{-1}\mathrm{M}_{-1}\mathrm{x} + \mathrm{M}\left(\mathrm{x}, t\right) \quad [\mathrm{M}_{-2}, \mathrm{M}_{-1}] = 0$$

- Show that for asmyptotic **monopole mass** $F_3\left(\infty\right) > 1$, can deform **abelian solution** $F_1 = 0$ in a complete two-parameter family.

Strategy Of Proof

- Extend **incompleteness** results of Lotay-Oliveria.
- Parameterise **solutions at infinity** using theory of **irregular singular initial value problems** of the form:

$$\dot{\mathrm{x}} = t^{-2}\mathrm{M}_{-2}\mathrm{x} + t^{-1}\mathrm{M}_{-1}\mathrm{x} + \mathrm{M}(\mathrm{x}, t) \quad [\mathrm{M}_{-2}, \mathrm{M}_{-1}] = 0$$

- Show that for asymptotic **monopole mass** $F_3(\infty) > 1$, can deform **abelian solution** $F_1 = 0$ in a complete two-parameter family.
- Use with **comparison** results: construct **closure** of complete solution set using deformed abelian solutions.

Asymptotically Conical (AC) limit

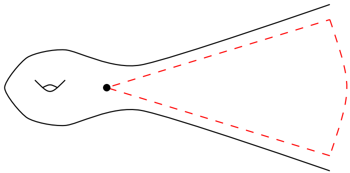
- $\mathbb{B}_7$-family of G_2-metrics g_ℓ (fixed scale) on $S^3 \times \mathbb{R}^4$. Setting $\ell = \infty$ gives **Bryant-Salamon** G_2-manifold.

Asymptotically Conical (AC) limit

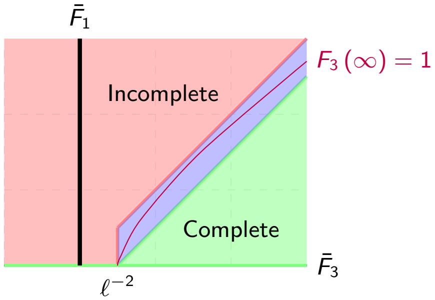
- $\mathbb{B}_7$-family of G_2-metrics g_ℓ (fixed scale) on $S^3 \times \mathbb{R}^4$. Setting $\ell = \infty$ gives **Bryant-Salamon** G_2-manifold.
- $(A_1, A_3, B_1, B_3) \sim \left(\frac{\sqrt{3}}{3}t, \frac{2}{3}t, \frac{\sqrt{3}}{3}t, \frac{2}{3}t\right)$ as $t \to \infty$.

Asymptotically Conical (AC) limit

- $\mathbb{B}_7$-family of G_2-metrics g_ℓ (fixed scale) on $S^3 \times \mathbb{R}^4$. Setting $\ell = \infty$ gives **Bryant-Salamon** G_2-manifold.
- $(A_1, A_3, B_1, B_3) \sim \left(\frac{\sqrt{3}}{3}t, \frac{2}{3}t, \frac{\sqrt{3}}{3}t, \frac{2}{3}t\right)$ as $t \to \infty$.
- $g \sim dt^2 + t^2 g_6$, G_2-cone metric.



Asymptotically Conical (AC) limit



- Curve $F_3(\infty) = 1$ (*) converging to **Clarke** solution $F_1 = F_3$ on Bryant-Salamon.

ALF-fibration

- Two-parameters (r_0, ℓ) define functions (A_1, A_3, B_1, B_3).

ALF-fibration

- Two-parameters (r_0, ℓ) define functions (A_1, A_3, B_1, B_3).
- **Adiabatic limit**: pulling back metric by a diffeomorphism $s_{\lambda^2} : t \mapsto \lambda^2 t$ and **rescaling**.

ALF-fibration

- Two-parameters (r_0, ℓ) define functions (A_1, A_3, B_1, B_3).
- **Adiabatic limit**: pulling back metric by a diffeomorphism $s_{\lambda^2} : t \mapsto \lambda^2 t$ and **rescaling**.
- Let $A_i^\lambda(t) := \lambda^{-2} A_i(\lambda^2 t), B_i^\lambda(t) := \lambda^{-1} B_i(\lambda^2 t)$. Then:

$$s_{\lambda^2}^* g = \lambda^4 \left(dt^2 + \left(A_i^\lambda\right)^2 \left(\sum_{i=1}^{3} (e_i + e_i')^2 \right) + \lambda^{-2} \left(B_i^\lambda\right)^2 \left(\sum_{i=1}^{3} (e_i - e_i')^2 \right) \right)$$

ALF-fibration

- Two-parameters (r_0, ℓ) define functions (A_1, A_3, B_1, B_3).
- **Adiabatic limit**: pulling back metric by a diffeomorphism $s_{\lambda^2} : t \mapsto \lambda^2 t$ and **rescaling**.
- Let $A_i^\lambda(t) := \lambda^{-2} A_i(\lambda^2 t), B_i^\lambda(t) := \lambda^{-1} B_i(\lambda^2 t)$. Then:

$$s_{\lambda^2}^* g = \lambda^4 \left(dt^2 + \left(A_i^\lambda\right)^2 \left(\sum_{i=1}^{3} (e_i + e_i')^2 \right) + \lambda^{-2} \left(B_i^\lambda\right)^2 \left(\sum_{i=1}^{3} (e_i - e_i')^2 \right) \right)$$

- Let $\lambda = r_0 \to 0$, then

$$B_i^\lambda \to 2 \qquad A_1^\lambda \to r^{\frac{1}{2}} \left(r - \tfrac{1}{\ell^2}\right)^{-1} \qquad A_3^\lambda \to r^{-\frac{1}{2}}$$

where $\dot{r} = -r^{-\frac{1}{2}} \left(r - \frac{1}{\ell^2}\right)^2$ giving the **Taub-NUT metric** on the fibre $\mathbb{R}^4$.

ALF-fibration

- Two-parameters (r_0, ℓ) define functions (A_1, A_3, B_1, B_3).
- **Adiabatic limit**: pulling back metric by a diffeomorphism $s_{\lambda^2} : t \mapsto \lambda^2 t$ and **rescaling**.
- Let $A_i^\lambda(t) := \lambda^{-2} A_i(\lambda^2 t), B_i^\lambda(t) := \lambda^{-1} B_i(\lambda^2 t)$. Then:

$$s_{\lambda^2}^* g = \lambda^4 \left(dt^2 + \left(A_i^\lambda\right)^2 \left(\sum_{i=1}^{3} (e_i + e_i')^2 \right) + \lambda^{-2} \left(B_i^\lambda\right)^2 \left(\sum_{i=1}^{3} (e_i - e_i')^2 \right) \right)$$

- Let $\lambda = r_0 \to 0$, then

$$B_i^\lambda \to 2 \qquad A_1^\lambda \to r^{\frac{1}{2}} \left(r - \tfrac{1}{\ell^2}\right)^{-1} \qquad A_3^\lambda \to r^{-\frac{1}{2}}$$

  where $\dot{r} = -r^{-\frac{1}{2}} \left(r - \frac{1}{\ell^2}\right)^2$ giving the **Taub-NUT metric** on the fibre $\mathbb{R}^4$.
- $s_{\lambda^2}^* g \sim \lambda^4 \left(g_{\mathrm{TN}} \oplus \lambda^{-2} g_{\mathrm{round}} \right)$

ALF-fibration

Theorem

(S.-Turner) There is a function $\bar{F}_i(r_0) \to \infty$, $\lambda \to 0$, *such that* $s^*_{\lambda^2}\mathbb{A}$ *in the limit* $\lambda = r_0 \to 0$ *gives a two parameter family of instantons on Taub-NUT fibred over* S^3, *one for each fixed* $\bar{F}_i r_0^4$.

- Moduli of invariant instantons on Taub-NUT is given by the same picture as for $\mathbb{B}_7$.

ALF-fibration

Theorem

(S.-Turner) There is a function $\bar{F}_i(r_0) \to \infty$, $\lambda \to 0$, *such that* $s^*_{\lambda^2}\mathbb{A}$ *in the limit* $\lambda = r_0 \to 0$ *gives a two parameter family of instantons on Taub-NUT fibred over* S^3, *one for each fixed* $\bar{F}_i r_0^4$.

- Moduli of invariant instantons on Taub-NUT is given by the same picture as for $\mathbb{B}_7$.
- Moduli of instantons on Taub-NUT is **hyperkahler mfld** (Cherkis et al.), in particular should be more instantons than the invariant ones we have found.

Summary

- Parametrised solutions to ODE system, geometric explination in terms of deforming abelian instantons on conifold.

Summary

- Parametrised solutions to ODE system, geometric explination in terms of deforming abelian instantons on conifold.
- Understood limiting behaviour of this solution space, using the two geometric limits (AC, ALF-fibration)

Summary

- Parametrised solutions to ODE system, geometric explination in terms of deforming abelian instantons on conifold.
- Understood limiting behaviour of this solution space, using the two geometric limits (AC, ALF-fibration)
- Suggestion for behaviour outside of the invariant setting

Summary

- Parametrised solutions to ODE system, geometric explination in terms of deforming abelian instantons on conifold.
- Understood limiting behaviour of this solution space, using the two geometric limits (AC, ALF-fibration)
- Suggestion for behaviour outside of the invariant setting

Thank You For Listening!