Complete G_2-solitons

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These slides available at
http://people.bath.ac.uk/jlpn20/g2sol3.pdf

Introduction

Context: Riemannian 7-manifolds with holonomy group G_2,
a special kind of Ricci-flat manifolds

Bryant's Laplacian flow: a cousin of Ricci flow for closed G_2-structures

G_2 solitons: self-similar solutions to Laplacian flow, equivalent to positive 3-form $\varphi \in \Omega^3(M^7)$, vector field X, constant $\lambda \in \mathbb{R}$ satisfying

$$d\varphi = 0, \quad \Delta_\varphi \varphi = \lambda \varphi + \mathcal{L}_X \varphi.$$

We have found asymptotically conical G_2 solitons of cohomogeneity one on $\Lambda^2_- \mathbb{C}P^2$ and $\Lambda^2_- S^4$, of all three types (shrinker, expander and steady), as well as complete solitons with different end behaviours.

Outline

1. Main results
2. Holonomy G_2 and Laplacian flow
3. Invariant soliton ODE and singular initial value problem
4. Forward-completeness
5. Twistor bundle perspective and rescaling limits of ODEs

1. Main results
Invariant G_2-structures on $\Lambda^2_- S^4$ and $\Lambda^2_- \mathbb{C}P^2$

$Sp(2)$-invariant G_2-structures φ on $\Lambda^2_- S^4 \setminus \text{zero section} \cong \mathbb{R}_{>0} \times \mathbb{C}P^3$
$\leftrightsquigarrow f_1, f_2 : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$
$\leftrightarrow$ scale of base and S^2 fibres of $\mathbb{C}P^3 \to S^4$.

$SU(3)$-invariant G_2-structures on $\Lambda^2_- \mathbb{C}P^2 \setminus \text{zero section} \cong \mathbb{R}_{>0} \times SU(3)/T^2$
$\leftrightsquigarrow f_1, f_2, f_3 : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$
$\leftrightarrow$ scale of S^2 fibres of three different fibrations $SU(3)/T^2 \to \mathbb{C}P^2$.

$f_2 = f_3 \Leftrightarrow$ multiplication by -1 on fibres of $\Lambda^2_- \mathbb{C}P^2$ is isometry.
ODEs identical for $SU(3) \times \mathbb{Z}_2$-invariant and $Sp(2)$-invariant solitons (indeed, same ODE appears on $\Lambda^2_- X$ for any positive Einstein self-dual X).

Cones: G_2-structure φ_C defined by $f_i = c_i r$.

Closed cones: $d\varphi_C = 0 \Leftrightarrow$ scale-normalisation of c

Then $X = -\frac{\lambda r}{3}\frac{\partial}{\partial r} \Rightarrow \lambda\varphi_C + \mathcal{L}_X \varphi_C = 0$, while $\Delta\varphi_C$ is of lower order
$\rightsquigarrow$ For $\lambda \neq 0$, $\left(\varphi_C, -\frac{\lambda r}{3}\frac{\partial}{\partial r}\right)$ can be used as a model for an AC end

Unique holonomy G_2 cone: $f_1 = f_2 = f_3 = \frac{r}{2}$

Explicit asymptotically conical shrinkers

Shrinkers provide models for formation of singularities, and tend to be rare.

Theorem A
There is a complete $SU(3) \times \mathbb{Z}_2$-invariant shrinking G_2-soliton on $\Lambda^2_- \mathbb{C}P^2$, and an $Sp(2)$-invariant one on $\Lambda^2_- S^4$, defined by $\lambda = -1$ and

$$f_1 = r, \quad f_2 = f_3 = \sqrt{\frac{9}{4} + \frac{r^2}{4}}, \quad X = \left(\frac{t}{3} + \frac{4t}{9 + t^2}\right) \frac{\partial}{\partial r}.$$

This is asymptotically conical because $\frac{f_i}{r} \to c_i$ (for $c = (1, \frac{1}{2}, \frac{1}{2})$).

The asymptotic cone is characterised by $\lim_{r \to \infty} \frac{f_1}{f_2} = 2$.

Asymptotic rate is $\nu = -2$ because $\frac{f_i}{r} = c_i + O(r^{-2})$.

Conjecture: This is the unique complete $Sp(2)$-invariant shrinker.

Asymptotically conical expanders on $\Lambda^2_- S^4$

Expanders provide models for how the flow can smooth out a singularity.

Theorem B

Every complete $Sp(2)$-invariant expanding G_2-soliton on $\Lambda^2_- S^4$ is AC with rate -2.

Up to scale, there is precisely a 1-parameter family of such expanders.

Their asymptotic limits are distinct, bijecting with $(0,1)$ by $\lim_{r\to\infty} \frac{f_1}{f_2}$.

Keeping the scale fixed, the family can be parametrised by $\lambda > 0$.

Limit as $\lambda \to 0$ is the Bryant-Salamon G_2-manifold (which has $\lim_{r\to\infty} \frac{f_1}{f_2} = 1$), considered as a static G_2-soliton.

Remark

The asymptotic cone of the explicit AC shrinker on $\Lambda^2_- S^4$ ($\lim_{r\to\infty} \frac{f_1}{f_2} = 2$) does *not* match the cone of any AC $Sp(2)$-invariant expander ($\lim_{r\to\infty} \frac{f_1}{f_2} < 1$).

Asymptotically conical expanders on $\Lambda^2_-\mathbb{C}P^2$

Theorem B also yields a corresponding 1-parameter family of $SU(3)\times\mathbb{Z}_2$-invariant expanders on $\Lambda^2\mathbb{C}P^2$.

While AC shrinker ends are rigid, AC expander ends are stable.

Perturbing 1-parameter family of $SU(3)\times\mathbb{Z}_2$-invariant solutions $\rightsquigarrow$

Theorem C

Up to scale, $\Lambda^2_-\mathbb{C}P^2$ admits a 2-parameter family of $SU(3)$-invariant expanding G_2-solitons that are AC with rate -2.

We do not expect *every* complete $SU(3)$-invariant expander to be AC.

$$f_1 = \frac{3}{\lambda r},\ f_2^2 = f_3^2 = r\, e^{\frac{\lambda r^2}{6}}$$

solves the soliton ODEs to leading order, and can be corrected to forward-complete solutions with doubly exponential volume growth.

Conjecture: The boundary of the 2-parameter family of $SU(3)$-invariant AC expanders corresponds to complete expanders with such ends.

Flowing through conical singularity?

If a singularity forms modelled on the explicit AC shrinker on $\Lambda^2_- S^4$, then no $Sp(2)$-invariant expander provides a model for how to smooth it out again.

Harder to control which closed $SU(3)$-invariant cones over $SU(3)/T^2$ appear as asymptotic limits of complete $SU(3)$-invariant expanders on $\Lambda^2_- \mathbb{C}P^2$, but numerics suggest:

the asymptotic cone of the shrinker on $\Lambda^2_- \mathbb{C}P^2$

does match

the asymptotic cone of some $SU(3)$-invariant expander on $\Lambda^2_- \mathbb{C}P^2$

after applying an order 3 automorphism to $SU(3)/T^2$ that does not extend to $\Lambda^2_- \mathbb{C}P^2$ (instead permuting 3 different S^2-fibrations $SU(3)/T^2 \to \mathbb{C}P^2$)

$\rightsquigarrow$ potential model for "flowing through" the singularity, crushing a $\mathbb{C}P^2$ and inflating it again in one of two topologically different ways.

This would realise a "G_2 conifold transition" (**Atiyah-Witten (2001)**).

Complete steady solitons

All known complete examples of steady Ricci solitons have sub-Euclidean volume growth. In contrast

Theorem D

There is precisely a 1-parameter family of $SU(3)$-invariant AC steady G_2 solitons on $\Lambda^2_-\mathbb{C}P^2$, all asymptotic with rate -1 to the unique $SU(3)$-invariant torsion-free cone ($f_i = \frac{r}{2} + O(1)$).

One limit is again the static soliton on the Bryant-Salamon AC G_2-manifold.

The other limit is an explicit complete steady G_2-soliton:

$$f_1 = \sqrt{1+e^{-r}},\ f_2 = \sqrt{1+e^{r}},\ f_3 = 2\sinh\frac{r}{2}, \quad X = \tanh\frac{r}{2}\frac{\partial}{\partial r}$$

Asymptotic geometry:

S^2 fibres in one other fibration $SU(3)/T^2 \to \mathbb{C}P^2$ have constant size.

Base is the sinh cone over $\mathbb{C}P^2$, i.e. the negative Einstein metric

$$dr^2 + (\sinh r)^2 g_{\mathbb{C}P^2} \quad \text{on} \quad \mathbb{R}_+ \times \mathbb{C}P^2.$$

2. Holonomy G_2 and Laplacian flow
Riemannian holonomy G_2

$G_2 := \mathrm{Aut}\ \mathbb{O}$, $\quad \mathbb{O} =$ octonions, normed division algebra of real dimension 8.

G_2 acts on $\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$, preserving metric, orientation, cross product

G_2 is the stabiliser in $GL(7, \mathbb{R})$ of a stable $\varphi_0 \in \Lambda^3(\mathbb{R}^7)^*$
(*i.e.* the $GL(7, \mathbb{R})$-orbit of φ_0 is open).

$\varphi \in \Omega^3(M^7)$ pointwise equivalent to φ_0 defines a G_2-structure.

Because $G_2 \subset SO(7)$, such a φ induces a metric and orientation.

$Hol(M) \subseteq G_2 \Leftrightarrow$ metric induced by some G_2-structure φ such that $\nabla\varphi = 0$.
Then call φ *torsion-free*. This is equivalent to the first-order non-linear PDE

$$d\varphi = d^*\varphi = 0.$$

Metrics with holonomy G_2 are always Ricci-flat.

All known constructions of examples on closed manifolds (**Joyce 1994**...)
solve the elliptic PDE by gluing together pieces with dimensional reduction.

Bryant-Salamon AC G_2 metrics

Most complete non-compact examples have a symmetry reduction.

First complete examples due to **Bryant-Salamon (1987)**. These

- are asymptotically conical, *i.e.* $M \setminus (\text{compact set}) \cong \mathbb{R}_+ \times \Sigma^6$ and

$$g = dr^2 + g_\Sigma + O(r^\nu), \quad \varphi = r^2 dr \wedge \omega + r^3 \alpha + O(r^\nu)$$

  for $\omega \in \Omega^2(\Sigma^6),\ \alpha \in \Omega^3(\Sigma^6)$ defining $SU(3)$-structure.

- have cohomogeneity 1 G-action, *i.e.* generic orbit Σ has dimension 6.

M	G	Σ	ν
$\Lambda^2_- S^4$	$Sp(2)$	$\mathbb{C}P^3$	-4
$\Lambda^2_- \mathbb{C}P^2$	$SU(3)$	$SU(3)/T^2$	-4
$S^3 \times \mathbb{R}^4$	$SU(2)^3$	$S^3 \times S^3$	-3

Last two Σ have $\mathbb{Z}_3$ of automorphisms that do not extend to diffeomorphisms of M

$\rightsquigarrow$ G_2 conifold transitions: 3 topologically distinct ways to glue in zero section to resolve conical singularity $\mathbb{R}_{>0} \times \Sigma$.

Bryant's Laplacian flow

Solve
$$\frac{d\varphi_t}{dt} = \Delta_{\varphi_t}\varphi_t$$
with initial condition φ_0 satisfying $d\varphi_0 = 0$. (Then $d\varphi_t = 0$ for all t.)

- Stationary points are exactly torsion-free G_2-structures.
- Gradient flow for $\mathrm{vol}(\varphi)$ restricted to cohomology class of φ_0.
- Induced metric evolves by

$$\frac{dg_t}{dt} = -2\mathrm{Ric}(g_t) + \text{ terms quadratic in torsion of } \varphi_t$$

Theorem (Bryant-Xu 2011, Lotay-Wei 2017)

Short-time existence and uniqueness.
Torsion-free G_2-structures are stable.

What is long term behaviour?? Expect singularities to form in finite time.
By analogy with other flows, expect solitons as models.

G_2 soliton equations

G_2-structure φ, vector field X, dilation constant $\lambda \in \mathbb{R}$ satisfying

$$\begin{cases} d\varphi = & 0, \\ \Delta_\varphi \varphi = & \lambda\varphi + \mathcal{L}_X \varphi. \end{cases}$$

$\Leftrightarrow$ self-similar solution of Laplacian flow

$$\varphi_t = k(t)^3 f^* \varphi, \qquad \frac{df}{dt} = k(t)^{-2} X, \qquad k(t) = \frac{3 + 2\lambda t}{3}$$

$\lambda > 0$: expanders (immortal solutions)
$\lambda = 0$: steady solitons (eternal solutions)
$\lambda < 0$: shrinkers (ancient solutions)

- Non-steady soliton $\Rightarrow \varphi$ exact
- Solitons on a compact manifold are stationary or expanders
- Scaling behaviour: (φ, X) is a λ-soliton $\Leftrightarrow (k^3\varphi, k^{-1}X)$ is a $k^{-2}\lambda$-soliton.

3. Initial value problem for invariant solitons
Invariant G_2-structures

$SU(3)$-invariant G_2-structures φ on $\mathbb{R}_+ \times SU(3)/T^2$ such that

- $\|\frac{\partial}{\partial r}\| = 1$, and
- restriction to each slice $SU(3)/T^2$ is closed

are parametrised by triples of functions $f_1, f_2, f_3 : \mathbb{R}_+ \to \mathbb{R}_+$.

$$\varphi = dr \wedge (f_1^2\omega_1 + f_2^2\omega_2 + f_3^2\omega_3) + f_1 f_2 f_3 \alpha$$

for $\omega_i \in \Omega^2(SU(3)/T^2)$ and $\alpha \in \Omega^3(SU(3)/T^2)$ $SU(3)$-invariant.

$f_i =$ scale of S^2 fibres of one of three possible fibrations $SU(3)/T^2 \to \mathbb{C}P^2$

$\mathrm{Vol}(SU(3)/T^2)$ proportional to $(f_1 f_2 f_3)^2$

Ones with $f_2 = f_3$ have extra $\mathbb{Z}_2$ symmetry, and also define analogous $Sp(2)$-invariant G_2-structures on $\mathbb{R}_+ \times \mathbb{C}P^3$.
(Then $f_1 =$ scale of S^2 fibres of $\mathbb{C}P^3 \to S^4$, and $f_2 = f_3$ a scale of base.)

Closure and soliton ODE

$$d\varphi = 0 \;\Leftrightarrow\; 2\frac{d}{dr}(f_1 f_2 f_3) = f_1^2 + f_2^2 + f_3^2$$

Cones $\leftrightarrow$ $f_i = c_i r$ linear

Then $d\varphi = 0 \;\Leftrightarrow\; 6c_1 c_2 c_3 = c_1^2 + c_2^2 + c_3^2$ is a scale-normalisation:

Unique choice of "cone angle" to make a closed cone for each homothety class on $SU(3)/T^2$

$\rightsquigarrow$ 2-parameter family of closed cones

The soliton condition for $\varphi = f_1^2\omega_1 + f_2^2\omega_2 + f_3^2\omega_3 + f_1 f_2 f_3 \alpha$ and $X = u\frac{\partial}{\partial r}$ is naively a 2nd-order ODE system for (f_1, f_2, f_3, u) (with some constraints).

Useful to rewrite it as a 1st-order ODE in $(f_1, f_2, f_3, \tau_1, \tau_2, \tau_3)$, where τ_i capture the torsion by $d^*\varphi = \tau_1\omega_1 + \tau_2\omega_2 + \tau_3\omega_3$.

This is a system in 5 variables after taking into account that

$$d\varphi = 0 \;\Rightarrow\; \varphi \wedge d^*\varphi = 0 \;\Rightarrow\; \frac{\tau_1}{f_1^2} + \frac{\tau_2}{f_2^2} + \frac{\tau_3}{f_3^2} = 0.$$

Smooth extension problem

Suppose that $H \subset G$, that H acts on vector space V, and that the action is transitive on unit sphere in V, with stabiliser $K \subset H$.

Then think of the vector bundle $G \times_K V := (G \times V)/K \to G/K$ as

$$\text{zero section } G/K \quad \sqcup \quad \mathbb{R}_+ \times G/H.$$

To find complete structures on $G \times_K V$, first step is to ask:

Which solutions on $(0, \epsilon) \times G/H$ extend smoothly over G/K at $r = 0$?

Applying methods **Eschenburg-Wang (2000)**

- Identify conditions on $f_i : [0, \epsilon) \to \mathbb{R}_+$ to ensure smooth extension of $\varphi = f_1^2\omega_1 + f_2^2\omega_2 + f_3^2\omega_3 + f_1 f_2 f_3 \alpha$ from $(0, \epsilon) \times SU(3)/T^2$ to $\mathbb{C}P^2$:
  - f_1 odd with $f_1'(0) = 1$ (so that S^2 fibres shrink to zero at right rate)
  - f_2 and f_3 even with $f_2(0) = f_3(0) = b = \sqrt[4]{\mathrm{Vol}(\mathbb{C}P^2)} > 0$
- Then solve by power series.

Solutions to the soliton initial value problem

Proposition

For each $\lambda \in \mathbb{R}$, there is a 2-parameter family $\varphi_{b,c}$ of solutions to the G_2-soliton equation with dilation constant λ defined for small r that extend smoothly to (a neighbourhood of zero section in) $\Lambda^2_- \mathbb{C}P^2$.

Two scale-invariant parameters: λb^2 and $\frac{c}{b}$.

$\rightsquigarrow$ up to scale there are 2-parameter families of local expanders and shrinkers on $\Lambda^2_- \mathbb{C}P^2$, and 1-parameter family of local steady solitons.

The parameter b is $\sqrt[4]{\mathrm{Vol}(\mathbb{C}P^2)}$, while c controls the leading term in $f_2 - f_3$.

The subfamily $\varphi_{b,0}$ has $f_2 = f_3$, so

- has extra $\mathbb{Z}_2$-symmetry (multiplication by -1 on fibres of $\Lambda^2_- \mathbb{C}P^2$)
- also defines solution near zero section of $\Lambda^2_- S^4$.

$\rightsquigarrow$ up to scale there are 1-parameter families of local expanders and shrinkers on $\Lambda^2_- S^4$, and a unique local steady soliton.

The latter defines the static soliton on the Bryant-Salamon G_2-manifold.

Hence there are no non-trivial $Sp(2)$-invariant steady solitons on $\Lambda^2_- S^4$.

4. Forward-completeness
Scale decoupling and AC ends for steady solitons

The steady case $\lambda = 0$ has a very different character because the scale

$$g := \sqrt[3]{f_1 f_2 f_3} = \sqrt[6]{\mathrm{vol}(\Sigma)}$$

essentially decouples from the homothety class

$$\left(\frac{f_1}{g}, \frac{f_2}{g}, \frac{f_3}{g} \right)$$

The latter evolves in a surface under a 2nd order autonomous ODE
$\Leftrightarrow$ 1st order ODE in 4 parameters

Torsion-free cone $c_1 = c_2 = c_3 = \frac{1}{2}$ is unique fixed point, and stable

$\Rightarrow$ Solutions with $\frac{f_i}{g}$ bounded are asymptotic to the torsion-free cone.
Eigenvalues of linearisation at fixed point give rate -1.
Since $\varphi_{b,0}$ is AC (static Bryant-Salamon), $\varphi_{b,c}$ is AC too for c near 0.

Trichotomy for $SU(3)$-invariant steady ends

Proposition

Any initial condition for an $SU(3)$-invariant steady soliton on $\mathbb{R} \times SU(3)/T^2$ leads to one of the following behaviours forward in time (up to permuting f_i)

(i) *AC with rate -1 to torsion-free cone*

(ii) *Complete with exponential volume growth: $f_1 \to \frac{1}{k}$, while $f_2 \sim f_3 \sim e^{kr}$.*

(iii) $f_1 = O(\sqrt{t_* - t})$, $f_2, f_3 = O((t_* - t)^{-1/4})$ *near finite extinction time t_*.*

We can decide the type of each smoothly closing local solution $\varphi_{b,c}$ thanks to spotting an explicit solution of type (ii) corresponding to $\varphi_{\sqrt{2},3}$

$$f_1 = \sqrt{1+e^{-r}},\ f_2 = \sqrt{1+e^{r}},\ f_3 = 2\sinh\tfrac{r}{2}, \quad u = \tanh\tfrac{r}{2}.$$

Theorem D

For $\lambda = 0$, the local solution $\varphi_{b,c}$ is

(i) *AC for $\frac{c^2}{b^2} < \frac{9}{2}$.*

(ii) *Exponentially growing for $\frac{c^2}{b^2} = \frac{9}{2}$.*

(iii) *Incomplete for $\frac{c^2}{b^2} > \frac{9}{2}$.*

Non-steady AC ends

The scale does not decouple for $\lambda \neq 0$. On the contrary, scaling up any point in the phase space makes λ terms more dominant, causing

Proposition

Any $SU(3)$-invariant non-steady soliton with all ratios $\frac{f_i}{f_j}$ bounded in forward time is AC with rate -2.

Schematically, because λ has dimensions of length^{-2}, the other factor S of those terms has dimensions of length^2, and satisfies an equation

$$\frac{dS}{dt} = -\lambda\alpha S + \beta$$

where α and β have dimension of length, and α involves only the f_i (not $\frac{df_i}{dr}$)
$\rightsquigarrow$ S has a limit as $r \to \infty$, so r^{-2} relative to its dimensions of length^2
$\rightsquigarrow$ AC rate is -2

This behaviour is **stable** for $\lambda > 0$
unstable for $\lambda < 0$

AC end solutions

Proposition

For each $\lambda \neq 0$ and (c_1, c_2, c_3) such that $c_1^2 + c_2^2 + c_3^2 = 6c_1c_2c_3$ there exists

- *a unique AC end solution if $\lambda < 0$*
- *a 2-parameter family of AC end solutions if $\lambda > 0$*

asymptotic to the corresponding closed cone (i.e. $\frac{f_i}{r} \to c_i$).

Letting $u = r^{-2}$, the sign of λ becomes significant in an ODE analogous to

$$\frac{dx}{du} = \lambda \frac{x}{u^2} + \frac{x}{2u} + 1 \qquad \text{with } x \to 0 \text{ as } u \to 0.$$

$\lambda < 0$: Unique smooth solution $x(u) = \sqrt{u} \exp\left(\frac{-\lambda}{u}\right) \int_0^u \exp\left(\frac{\lambda}{s}\right) \frac{ds}{\sqrt{s}}$

$\lambda = 0$: General solution $x(u) = 2u + C\sqrt{u}$

$\lambda > 0$: Any two smooth solutions differ by multiple of $\sqrt{u} \exp\left(-\frac{\lambda}{u}\right)$

Haskins-Khan-Payne (2022) prove rigidity of AC ends for gradient shrinking G_2 solitons without cohomogeneity one assumption

AC shrinkers

Heuristic for $\lambda < 0$:

Invariant shrinkers on $\mathbb{R}_+ \times SU(3)/T^2$ are flow lines in 5-dim phase space.
In 4-dimensional space of flow lines

- 2-dimensional submanifold extends across zero section $\mathbb{C}P^2 \subset \Lambda^2_- \mathbb{C}P^2$
- 2-dimensional submanifold has AC behaviour

Expect isolated intersections $\rightsquigarrow$ *finitely* many AC shrinkers on $\Lambda^2_- \mathbb{C}P^2$.

Theorem A

For $\lambda = -1$, $\varphi_{\frac{3}{2},0}$ is the explicit solution

$$f_1 = r, \quad f_2^2 = f_3^2 = \frac{9}{4} + \frac{r^2}{4}, \quad u = \frac{r}{3} + \frac{4r}{9 + r^2}.$$

$\rightsquigarrow$ *$SU(3) \times \mathbb{Z}_2$-invariant shrinker on $\Lambda^2_- \mathbb{C}P^2$ and $Sp(2)$-invariant shrinker on $\Lambda^2_- S^4$, AC with rate -2 to cone with $c_1 = 1$, $c_2 = c_3 = \frac{1}{2}$.*

Conjecture: Unique complete shrinker with this symmetry.

Trichotomy for $Sp(2)$-invariant expander ends

In the expander case,

$$f_1 = \frac{3}{\lambda r}, \; f_2 = A\sqrt{r}\, e^{\frac{\lambda r^2}{12}}$$

can be uniquely corrected to a forward-complete end solution for each $A > 0$.

This 1-parameter family of end solutions with doubly exponential growth forms a "wall" between two different stable types of forward-evolution.

Theorem

Any initial condition for an $Sp(2)$-invariant expander on $\mathbb{R} \times \mathbb{C}P^3$ leads to one of the following behaviours forward in time

(i) *AC with rate -2 to a closed cone*

(ii) *Complete with doubly exponential growth: $f_1 \sim \frac{3}{\lambda r}$, while $f_2 \sim \sqrt{r} e^{\frac{\lambda r^2}{12}}$.*

(iii) *$f_1 = O(\sqrt{t_* - t})$, $f_2 = O((t_* - t)^{-1/4})$ near finite extinction time t_*.*

All smoothly-closing $Sp(2)$-invariant solutions $\varphi_{b,0}$ on $\Lambda^2_- S^4$ fall in case (i), but we expect an analogous transition to be relevant for smoothly-closing $SU(3)$-invariant expanders $\varphi_{b,c}$ on $\Lambda^2_- \mathbb{C}P^2$.

$Sp(2)$-invariant AC expanders on $\Lambda^2_- S^4$

Theorem B

(i) *Every $Sp(2)$-invariant local expander $\varphi_{b,0}$ defined near the zero section of $\Lambda^2_- S^4$ is AC.*

(ii) *Moreover $\mathcal{L} : b \mapsto \lim_{r\to\infty} \frac{f_1}{f_2}$ is a continuous bijection $(0,\infty) \to (0,1)$ (so each closed $Sp(2)$-invariant cone $f_i = c_i t$ such that $c_1 < c_2$ is the asymptotic cone of unique complete expander).*

Elementary techniques suffice for (i) and non-decreasingness of $\mathcal{L}$, checking that certain inequalities are preserved.

$\mathcal{L}$ *strictly* increasing builds on detailed understanding of ends with given asymptotic cone.

$b \to 0$ limit equivalent up to scale to keeping b fixed and letting $\lambda \to 0$. This limit is the Bryant-Salamon manifold (with $c_1 = c_2 = \frac{1}{2}$).

$b \to \infty$ limit requires a more subtle comparison.

5. Twistor bundles and rescaling limits of ODEs
Rescalings of the $Sp(2)$-invariant soliton ODE

Recall variables in $Sp(2)$-invariant soliton ODEs:

$f_2 \leftrightarrow$ scale of the base of $\Lambda^2_- S^4 \to S^4$

$f_1 \leftrightarrow$ scale of S^2 fibres when foliating by sphere bundles

Changing variable f_2 to $\epsilon f_2 \Leftrightarrow$ rescaling "reference metric" on base of $\Lambda^2_- S^4$

Taking $\epsilon \to 0$

$\rightsquigarrow$ ODE for solitons on $\Lambda^2_- \mathbb{R}^4$ invariant under $\mathrm{Isom}(\mathbb{R}^4) = SO(4) \ltimes \mathbb{R}^4$.

This limit ODE is easier to analyse due to additional scale invariance and a conserved quantity.

The expander system has an explicit solution $f_1 = \dfrac{3}{\lambda r}$, $f_2^2 = r\, e^{\frac{\lambda r^2}{6}}$, incomplete at $r = 0$ but helps understand transition in original system.

Limit of $\varphi_{b,0}$ as $b \to \infty \rightsquigarrow$ AC solution, which helps control $\mathcal{L}(b)$

Taking $\epsilon = \sqrt{-1}$

$\rightsquigarrow$ ODE for solitons on $\Lambda^2_- H^4$ invariant under $\mathrm{Isom}(H^4) = SO(4,1)$.

Twistor space interpretation

For X^4 oriented, consider on the sphere bundle in $\Lambda^2_- X$ (twistor space)

- volume form ω_1 on fibres
- tautological 2-form ω_2

For X self-dual Einstein, the conditions for the Isom(X)-invariant

$$\varphi = dr \wedge (f_1^2 \omega_1 + f_2^2 \omega_2) + f_1 f_2^2 d\omega_1 \in \Omega^3(\Lambda^2_- X)$$

to be a soliton depend only on the scalar curvature κ. So

$\kappa = 1 \rightsquigarrow Sp(2)$-invariant ODE on $\Lambda^2_- S^4$;
in general get the $\epsilon = \sqrt{\kappa}$ rescaling.

In particular, the $\epsilon = 0$ limit can be interpreted as ODE for warped products solitons on $\mathbb{R} \times S^2 \times Y^4$, with Y hyper-Kähler.

The steady case was considered by **Ball (2022)**.

Rescalings of the $SU(3)$-invariant soliton ODE

In a similar way, rescaling f_2 and f_3 in the $SU(3)$-invariant soliton ODEs is related to considering $\Lambda^2_- X$ for self-dual *Kähler*-Einstein X^4 for different κ.

(In particular, one could consider $SU(2,1)$-invariant solitons of $\Lambda^2_- \mathbb{C}H^2$).

$f_2 = f_3 \Leftrightarrow$ invariance under under multiplying fibres of $\Lambda^2_- X$ by -1, recovering ODE from the case not requiring Kähler.

The limit of rescalings of the explicit exponentially growing solution $f_1 = \sqrt{1+e^{-r}},\ f_2 = \sqrt{1+e^{r}},\ f_3 = 2\sinh\frac{r}{2}$ give an explicit solution

$$f_1 = 1,\ f_2 = f_3 = e^r$$

on $\mathbb{R}_{>0} \times S^2 \times Y$ for Y hyperKähler

$\rightsquigarrow$ complete steady G_2 soliton that is metric product of S^2 and $dr^2 + e^r g_Y$

Another rescaling of $Sp(2)$-invariant soliton ODE

Changing variables $f_1 \to \epsilon^2 f_1$, $f_2 \to \epsilon f_2$ and taking $\epsilon \to 0$

$\Leftrightarrow$ rescaling both fibres and base of S^2-fibration $\mathbb{C}P^3 \to S^4$ of link of $\Lambda^2_- S^4$

$\rightsquigarrow$ soliton ODE on $\mathbb{R}\times\mathbb{C}\times\mathbb{C}^2$, invariant under complex Heisenberg group $\mathcal{H}_3$.

Same as soliton ODE for certain G_2-structures on $\mathbb{R} \times$ suitable T^2-bundle over hyper-Kähler Y^4.

Fowdar (2022): Explicit shrinker solution $f_1 = 4e^r$, $f_2 = e^{\frac{r}{2}}$

We find another complete shrinker with one AC end and one cusp end modelled on Fowdar's.