Giada Franz: Minimal surfaces via eigenvalue optimisation

Overall plan: These lectures will explore the relationship between minimal surfaces and eigenvalue optimization. We will mainly focus on free boundary minimal surfaces and their connection to the Steklov eigenvalue problem. We will prove important basic properties and then study some recent results about the behavior of Steklov optimizers for surfaces with fixed genus and number of boundary components converging to infinity. We will see how, in this asymptotic regime, Steklov optimization is related to Laplace optimization and closed minimal surfaces.

Talk 1.1: Correspondence between eigenvalue optimisation and minimal surfaces

Abstract: In this first lecture, we will introduce the Laplace and Steklov eigenvalue problems on a surface (without or with boundary, respectively). Then, we will explain how these problems are related to minimal surfaces in spheres and free boundary minimal surfaces in balls. In particular, we will focus on the relationship between Steklov eigenvalue optimisation and free boundary minimal surfaces in the unit ball, with proofs and basic properties.

Literature: Sections 1 and 2 of [FS13], in particular proof of Proposition 2.4 therein.

- [FS13] A. Fraser and R. Schoen, *Minimal surfaces and eigenvalue problems*, Geometric analysis, mathematical relativity, and nonlinear partial differential equations, 105–121, Contemp. Math. **599** (2013).

Talk 1.2: Steklov eigenvalues and the topology of a surface

Abstract: In this talk, we will prove upper bounds for the first (renormalized) Steklov eigenvalue, and its multiplicity, in terms of the topology of the surface. This shows an interesting connection between analytic properties of a surface (i.e. the Steklov eigenvalues) and topological ones.

Literature: Section 2 in [FS16], in particular Theorems 2.2 and 2.3.

- [FS16] A. Fraser and R. Schoen, *Sharp eigenvalue bounds and minimal surfaces in the ball*, Invent. Math. **203** (2016), no. 3, 823–890.

Talk 1.3: Free boundary minimal surfaces in $\mathbb{B}^3$: area bounds, compactness, uniqueness

Abstract: In this third lecture, we will give an overview on properties of free boundary minimal surfaces in $\mathbb{B}^3$ whose proof relies on the study of the Steklov eigenvalue problem on the surfaces. In particular, we will discuss area bounds of a surface in terms of its topology, compactness of the space of free boundary minimal surfaces with a fixed topology, and uniqueness results for free boundary minimal annuli with reflection symmetries.

Literature: Proposition 3.4, Corollary 3.6, and Theorem 1.2 in [FL14]. Time permitting, proof of Theorem 3.1 (specialized to surfaces in $\mathbb{B}^3$). Theorem 6.6 in [FS16] and Corollary 1 in [McG18].

- [FL14] A. Fraser and M. M.-C. Li, *Compactness of the space of embedded minimal surfaces with free boundary in three-manifolds with nonnegative Ricci curvature and convex boundary*, J. Differential Geom. **96** (2014), no. 2, 183–200.
- [FS16] A. Fraser and R. Schoen, *Sharp eigenvalue bounds and minimal surfaces in the ball*, Invent. Math. **203** (2016), no. 3, 823–890.
- [McG18] P. McGrath, *A characterization of the critical catenoid*, Indiana Univ. Math. J. **67** (2018), no. 2, 889–897.

Talk 1.4: Relation between Steklov and Laplacian eigenvalue optimization

Abstract: In this talk, we will present a relation between Laplacian eigenvalue optimization on a closed surface and Steklov eigenvalue optimization on a subset with boundary of the initial surface. If time allows, we will see how this can be proven as a consequence of a min-max characterization of conformal eigenvalues by Karpukhin–Stern.

Literature: Theorem 1.6 in [KS24], Corollary 1.6 in [GL21]. Time permitting, proof of Theorem 1.6 in [KS24] in Section 5.3 of the same paper.

- [KS24a] M. Karpukhin and D. Stern, *Min-max harmonic maps and a new characterization of conformal eigenvalues*, J. Eur. Math. Soc. (JEMS) **26** (2024), no. 11, 4071–4129.
- [GL21] A. Girouard and J. Lagacé, *Large Steklov eigenvalues via homogenisation on manifolds*, Invent. Math. **226** (2021), no. 3, 1011–1056.

Talk 1.5: Behavior of Steklov optimisers for large number of boundary components

Abstract: In this talk, we will show that free boundary minimal surfaces in the ball, obtained from maximisation of the first Steklov eigenvalue, converge to minimal surfaces in the sphere, as the number of boundary components converge to infinity and the genus is fixed. This gives an interesting relationship between free boundary minimal surfaces in balls and minimal surfaces in spheres. The most explicit example of this behavior are a sequence of free boundary minimal surfaces with genus zero in $\mathbb{B}^3$, which converge to the sphere $\mathbb{S}^2$.

Literature: Theorem 1.1 in [KS24b] (potentially with idea of the proof, Section 3 therein, or related results). Lemma 2.1 and Theorem 1.5 in [MZ24].

- [KS24b] M. Karpukhin and D. Stern, *From Steklov to Laplace: free boundary minimal surfaces with many boundary components*, Duke Math. J. **173** (2024), no. 8, 1557–1629.
- [MZ24] P. McGrath and J. Zou, *On the areas of genus zero free boundary minimal surfaces embedded in the unit 3-ball*, J. Geom. Anal. **34** (2024), no. 9, Paper No. 274, 14.

Talk 1.6 (by the mentor): Recap, recent results and future directions

Abstract: In this final lecture, we will give a brief recap, and present recent results and future directions in the field. In particular, we will focus on recent papers about existence of maximising metrics for the first (renormalized) Laplace eigenvalue, and existence of many new examples of minimal surfaces in $\mathbb{S}^3$ and free boundary minimal surfaces in $\mathbb{B}^3$ by eigenvalue optimisation.

Literature:

- [KKMS] M. Karpukhin, R. Kusner, P. McGrath, and D. Stern, *Embedded minimal surfaces in $\mathbb{S}^3$ and $\mathbb{B}^3$ via equivariant eigenvalue optimization*, preprint, available at `arXiv:2402.13121`.
- [Pet] R. Petrides, *Geometric spectral optimization on surfaces*, preprint, available at `arXiv:2410.13347`.
- [KPS] M. Karpukhin, R. Petrides, and D. Stern, *Existence of metrics maximizing the first Laplace eigenvalue on closed surfaces*, preprint, available at `arXiv:2505.05293`.

Stephen Lynch: Self-similar shrinkers and mean curvature flow

Overall plan: The mean curvature flow is intimately connected with minimal surface theory. For starters, it is formally a gradient flow for the area functional, and its equilibrium states are minimal surfaces. Perhaps more profoundly, parabolic rescalings of the flow often converge to self-similarly shrinking surfaces, which are minimal surfaces in Gaussian space. These lectures will revolve around aspects of the mean curvature flow theory which parallel or have been inspired by minimal surface theory. In particular, we will see how classification results for self-shrinkers have led to major recent breakthroughs in our understanding of the flow beyond singularities.

Talk 2.1: Self-shrinkers as singularity models

Abstract: We will introduce the mean curvature flow, describe the phenomenon of finite-time singularity formation and provide some key examples. Then we will derive one of the most important tools used to analyse singularities, Huisken's monotonicity formula. Finally, we will discuss Ilmanen's proof that central blow-ups are invariant under parabolic rescalings; that is, they are self-shrinkers.

Literature:

- Andrews, B., Chow, B., Guenther, C., and Langford, M. (2022). Extrinsic geometric flows (Vol. 206). American mathematical society. *(Background on mean curvature flow, derivation of Huisken's monotonicity formula.)*
- G. Huisken, Asymptotic behavior for singularities of the mean curvature flow, J. Differential Geom. **31** (1990), no. 1, 285–299. *(Derivation of Huisken's monotonicity formula.)*
- T. Ilmanen, Singularities of mean curvature flow of surfaces. preprint (1995). Available: https://people.math.ethz.ch/ ilmanen/papers/sing.ps. *(Section 2 proves that central blow-ups are self-shrinkers.)*

Talks 2.2: Ilmanen's local Gauss–Bonnet estimate and consequences.

Abstract: One of the key features of immersed surfaces, as opposed to submanifolds of larger dimension, is that the Gauss–Bonnet formula can be leveraged to get analytic control via total curvature bounds. In this lecture we will discuss a powerful implementation of this idea, in the form of Ilmanen's local Gauss–Bonnet estimate. We will then discuss closely related ε-regularity theorems for the mean curvature flow and minimal surfaces. This discussion is leading towards a proof that blow-ups of a mean curvature flow of dimension two are *smooth* in the next talk.

Literature:

- T. Ilmanen, Singularities of mean curvature flow of surfaces. preprint (1995). Available: https://people.math.ethz.ch/ ilmanen/papers/sing.ps. *(Section 1 proves the local Gauss-Bonnet estimate, Section 5 the ε-regularity theorem.)*
- Colding, T. H., and Minicozzi, W. P. (2011). A course in minimal surfaces (Vol. 121). American Mathematical Soc.. *(See Section 2 of Chapter 2 for a related ε-regularity theorem for minimal surfaces.)*

Talk 2.3 Surfaces with bounded total curvature: concentration and compactness.

Abstract: In a fundamental 1985 paper, Choi and Schoen studied sequences of compact minimal surfaces with bounded genus in a Riemannian ambient space. They proved that such sequences converge (in the varifold sense) to a smooth minimal surface and, moreover, the convergence is smooth away from finitely many points. (Under the extra hypothesis that the ambient space has positive Ricci curvature, the convergence is actually smooth everywhere and occurs with multiplicity one, hence compact minimal surfaces of fixed genus are compact in the smooth topology.) In this talk we will discuss their proof, and how similar ideas could be used by Ilmanen to prove that central blow-ups are smooth for mean curvature flows of dimension 2.

Literature:

- H. I. Choi and R. M. Schoen, The space of minimal embeddings of a surface into a three-dimensional manifold of positive Ricci curvature, Invent. Math. **81** (1985), no. 3, 387–394.
- T. Ilmanen, Singularities of mean curvature flow of surfaces. preprint (1995). Available: https://people.math.ethz.ch/ ilmanen/papers/sing.ps. *(Smoothness of blow-ups is proved in Section 4.)*

Talk 2.4: Classification of shrinkers with positive mean curvature.

Abstract: To the largest extent possible, we would like to classify self-shrinkers. This is a fascinating problem in its own right, which also reveals the possible shapes of singularities of the mean curvature flow and therefore has far-reaching consequences in geometry. Huisken took a big step in this endeavour by demonstrating that the round spheres and cylinders are the *only* self-shrinkers with *positive mean curvature*. This talk will cover his proof and a slight generalisation due to Colding and Minicozzi.

Literature:

- G. Huisken, Asymptotic behavior for singularities of the mean curvature flow, J. Differential Geom. **31** (1990), no. 1, 285–299. *(Compact mean convex shrinkers are classified in Section 4.)*
- G. Huisken, Local and global behaviour of hypersurfaces moving by mean curvature, in *Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990)*, 175–191, Proc. Sympos. Pure Math., 54, Part 1, Amer. Math. Soc., Providence, RI. *(In this paper Huisken extends his method to noncompact mean-convex shrinkers with bounded second fundamental form.)*
- T. H. Colding and W. P. Minicozzi II, Generic mean curvature flow I: generic singularities, Ann. of Math. (2) **175** (2012), no. 2, 755–833. *(Here the assumption of bounded second fundamental form is removed, in Section 10.)*

Talk 2.5: Classification of genus-zero shrinkers.

Abstract: This talk continues our discussion of the classification problem for self-shrinking surfaces, focusing on Brendle's beautiful classification of self-shrinkers of genus zero.

Literature:

- S. Brendle, Embedded self-similar shrinkers of genus 0, Ann. of Math. (2) **183** (2016), no. 2, 715–728.

Talk 2.6 (by the mentor): Recent breakthroughs and open problems in mean curvature flow.

Abstract: We will discuss how all of the results of the previous talks have played a pivotal role in major recent breakthroughs in the study of mean curvature flow. Particular emphasis will be placed on consequences of the Multiplicity One Conjecture recently resolved by Bamler–Kleiner, and on genericity results for singularity models by Chodosh–Choi–Mantoulidis–Schulze. This discussion will bring us to the forefront of current research, where numerous open questions and challenges remain.

Paul Minter: Frequency Functions in Minimal Surface Theory

Overall plan: The focus of these lectures is to investigate the use of Almgren's *frequency function* as a tool for controlling the size of the singular set for minimal surfaces. Loosely speaking, at singular points where the tangent space occurs with some integer multiplicity ≥ 2 (known as *branch points*), the frequency function is an (almost) monotone quantity which captures the rate of decay of the minimal surface to its tangent plane. The monotonicity property enables one to analyse such singular points using stratification techniques analogously to other geometric variational problems, however numerous additional subtleties occur in this setting due to the possibility of multiplicity, which can cause singularities to disappear or not concentrate well in blow-up limits. Indeed, it is not even know if one can always introduce a frequency function in general, let alone prove its almost monotonicity!

To warm us up, we will introduce the concept of the frequency function in the simplest possible setting: harmonic functions $u : B_1^n(0) \to \mathbb{R}$. This will allow us to get acquainted with the basic ideas, and we will see how one can use the frequency function to control the size and structure of the touching set of two harmonic functions. For this reason, the frequency function can be seen as a tool which provides *quantitative* unique continuation results. Next, we will generalise these ideas to more general quasilinear elliptic PDEs, including the minimal surface equation. Up to this point this should build upon the PDE knowledge people have.

Next, we will introduce the concept of a *multi-valued function*, as defined by Almgren, and will look at the theory of multi-valued minimisers of the Dirichlet energy (called *Dir-minimisers*). These objects exhibit the branch point singularities mentioned previously for minimal surfaces, and indeed Dir-minimisers are the correct linearised object to study area-minimising surfaces. Our primary concern here will be to understand the size of the branch set of a Dir-minimiser, and we will see how frequency functions can be used in this setting to achieve this. After all of this, we will then move to discussing how one can try to adapt these ideas to branched minimal surfaces. For this we will look at the simplest possible setting, where the minimal surface is given by a 2-valued $C^{1,\alpha}$ graph, and introduce the newly discovered notion of *planar frequency*, used by B Krummel & N. Wickramasekera in their work on area-minimising currents. We will then end by discussing the final hurdle one needs to overcome, and how a geometric viewpoint of the frequency function leads to the idea of a center manifold to resolve this.

Talk 3.1: Introduction to Frequency Functions I: Harmonic Function Case Study

Abstract: The aim of this lecture is to introduce the notion of the frequency function as a tool for quantitative unique continuation, using harmonic functions as a case study. We will see how the monotonicity of the frequency function follows from two key integral identities (known as *squash* and *squeeze*, coming from outer and inner variations respectively). Next, we will use this to control the Hausdorff dimension of both the zero-set of a harmonic function as well as its *critical*-set. This extends the well known "strong" unique continuation theorem for harmonic functions to the following statement: if $u : B_1^n(0) \to \mathbb{R}$ is harmonic and vanishes on a set A with $\mathcal{H}^{n-1+\gamma}$-positive measure, then necessarily $u \equiv 0$. As a consequence, we can control the size of the coincidence set of two harmonic functions.

Pre-requisites: This lecture should only require a basic knowledge of elliptic PDE theory.

Literature: The literature on these results are somewhat scattered, and a clear reference for these specific results does not immediately come to mind. The person giving this lectures should get in touch with me and I can help direct them.

Talk 3.2: Introduction to Frequency Functions II: Quasilinear Elliptic PDEs

Abstract: The aim of this lecture is to now generalise the ideas from the first talk to more general quasilinear elliptic PDEs. Slightly different results can also be emphasised, in particular how monotonicity of the frequency function gives rise to certain "doubling conditions", which lead to a "weak" unique continuation theorem. The key difference here is that the quasilinear nature of the PDE forces one to modify the form of the frequency function into one which reflects the structure of the PDE, a phenomenon which will be important later for 'planar frequency'. A particular emphasis can be given to the minimal surface equation if desired.

Pre-requisites: This lecture should only require a basic knowledge of elliptic PDE theory, perhaps some familiarity of quasilinear elliptic PDEs would help.

Literature: The main reference for this lecture is the following:

- *N. Garofalo and F. Lin, Monotonicity properties of variational integrals, A_p weights and unique continuation, Indiana Univ. Math. J.* **35** *(1986), no. 2, 245–268.*

From this, we want to see Theorem 1.2 and Theorem 1.3.

Talk 3.3: Introduction to Multi-Valued Functions and Dirichlet Minimisers

Abstract: We now change gears and turn to multi-valued functions, as introduced by Almgren. This framework allows one, for instance, to make sense of the complex square-root $z^{1/2}$, as a two-valued function. We will introduce the space $\mathcal{A}_Q(\mathbb{R}^k)$ of unordered Q-tuples of points in $\mathbb{R}^k$ and endow it with a metric which allows us to talk about continuity, Lipschitz regularity, $W^{1,2}$, and so-forth for multi-valued functions. The notion of a mutli-valued function minimising the Dirichlet energy will then be introduced, and the question of its regularity raised.

Pre-requisites: This lecture should not require much pre-requisite knowledge outside of a standard course in functional analysis.

Literature: The main reference for this lecture is the following:

- *C. De Lellis and E. Spadaro, Q-valued functions revisited, Vol. 211. No. 991. American Mathematical Society, 2011.*

From this, we want to see Chapters 1 and 2.

Talk 3.4: The Singular Set of a Dirichlet Minimiser

Abstract: Building upon the material from the previous lecture, we will see how one can introduce a frequency function for Dirichlet minimisers analogously to how we did for harmonic functions. We will then use the frequency function to control the size of the singular set of the Dirichlet minimiser, in an analogous manner to how we controlled the touching set of two harmonic functions.

Pre-requisites: As this lecture builds upon Lecture 3 one would need to be also familiar with that material, although it shouldn't be too hard to pick it up.

Literature: The main reference for this lecture is the following:

- *C. De Lellis and E. Spadaro, Q-valued functions revisited, Vol. 211. No. 991. American Mathematical Society, 2011.*

From this, we want to see Chapter 3, in particular Proposition 3.22.

Talk 3.5: Branch Sets of Minimal Surfaces and Planar Frequency

Abstract: We now want to apply the ideas of the lectures up until this point to branched minimal surfaces. We should do this in the simplest possible setting for concreteness, which would be when the minimal surface is given by the graph of a 2-valued $C^{1,\alpha}$ function (arbitrary dimension and codimension). Assuming a suitable a priori estimate holds when the L^2 norm is small, we will show almost-monotonicity of the *planar frequency* function introduced by Krummel–Wickramasekera. This can be done simplest in codimension one, and we can stick with that setting if desired. We will see how this allows us to control the size of a significant piece of the branch set, but unfortunately not necessarily *all* of it. This will make clear the difference between the linear situation of Dirichlet minimisers and the non-linear, geometric setting of minimal surfaces. As an aside, some mention of how this problem can be handled via other means, *specifically in the 2-valued setting*, following the work of Simon–Wickramasekera can be discussed.

Pre-requisites: Uses of frequency functions up until this point of the lectures, some experience with geometric measure theory would be helpful.

Literature: The main references for this lecture are the following:

- *B. Krummel and N. Wickramasekera, Analysis of singularities of area minimizing currents: planar frequency, branch points of rapid decay, and weak locally uniform approximation, arXiv preprint arXiv:2304.10653 (2023).*
- *L. Simon and N. Wickramasekera, A frequency function and singular set bounds for branched minimal immersions, Communications on Pure and Applied Mathematics 69.7 (2016): 1213-1258.*

What I exactly have in mind has not appeared in the literature (yet! coming soon...) in the form I am imagining, so the person giving this lecture should get in touch with me and I can help direct them.

Talk 3.6 (by the mentor): Geometric Frequency and Center Manifolds (in light of Planar Frequency)

Abstract: In this final lecture, we will discuss how one gets around the final hurdle described in Lecture 5, namely controlling the size of the set of branch points where the planar frequency is exactly 2. This will involve the construction of a new object which we can measure frequency with respect to, known as a *center manifold*. Using planar frequency, we are able to drastically simplify this construction. We won't have time to talk about all the details as nonetheless it is still a lengthy construction, but we should be able to give a good idea. Then we will finish by discussing the current state of affairs for this problem and controlling the singular set of stationary integral varifolds in general.

Literature: The main references for this lecture are the following:

- *C. De Lellis and E. Spadaro, Regularity of area minimizing currents II: center manifold, Annals of Mathematics (2016): 499-575.*
