Melanie Graf: Singularity theorems in low regularity

Overall plan: In recent years, low regularity analytic methods have become increasingly important within Mathematical General Relativity, allowing us to treat non-smooth physical models in a rigorous mathematical framework. One area where this has been especially relevant is the study of the so-called singularity theorems. The classical singularity theorems of R. Penrose and S. Hawking from the 1960s are beautiful examples of mathematical results in Lorentzian Geometry with wide physical relevance for General Relativity showing that any spacetime with a smooth Lorentzian metric satisfying certain energy conditions and causality assumptions must be geodesically incomplete and the importance of establishing low regularity versions has already been pointed out in the classical textbook of Hawking and Ellis. During this series of talks we will cover recent results on singularity theorems for metrics that are merely C^1. Along the way we will explore questions such as: What is the minimal regularity of the metric required for a weak formulation of Einsteins equations? How to best approximate a given spacetime? And how do geodesics and causality theory behave in low regularity?

The main references for this are:

- [S] R Steinbauer: The Singularity Theorems of General Relativity and Their Low Regularity Extensions, `https://arxiv.org/abs/2206.05939`
- [G] M. Graf: Singularity theorems for C^1-Lorentzian metrics, `https://arxiv.org/abs/1910.13915`
- [KOSS] M Kunzinger, A Ohanyan, B Schinnerl, R Steinbauer: The Hawking-Penrose Singularity Theorem for C^1-Lorentzian Metrics, `https://arxiv.org/abs/2110.09176`, may be helpful as it contains a summary of [G] in section 2 and the presentation of distributions and relevant ODE theory results is less "ad hoc" than in [G].

Talk 1.1: The classical singularity theorems

Abstract: In this introductory lecture we will review some techniques from semi-Riemannian geometry concerning focal points/conjugate points and present the ideas for the proof of the classical singularity theorems of Hawking and of Penrose.

Literature: Sections 2 and 3 (minus the part concerning the Hawking-Penrose Theorem) of [S] and references therein. The following books also do a good job in treating the singularity theorems from a mathematical perspective:

- [ON] O'Neill: Semi-Riemannian Geometry With Applications to Relativity. In particular chapters 10 (background on conjugate and focal points via index form methods) and 14 (specifically Theorems 55A, 55B and 61 in chapter 14).
- [BEE] Beem, Ehrlich, Easley: Global Lorentzian Geometry. In particular chapters 11 and 12.

Talk 1.2: Distributions on manifolds and the weak formulation of Einsteins equations

Abstract: In the second talk we will introduce the theory of distributions on manifolds. We will see that due to the non linear dependence of the curvature on the metric there is a specific minimal regularity we need to demand of a Lorentzian metric to allow us to rigorously understand the Einstein equations in a weak, i.e. distributional, sense.

Literature: Sections 4.1 to 4.3 of [S]. Additionally

- [LM] LeFloch, P., Mardare, C.: Definition and stability of Lorentzian manifolds with distributional curvature, `https://arxiv.org/abs/0712.0122`. Sections 1-4, without 3.3, 4.3 and 4.4
- and for background [GKOS] Grosser, M., Kunzinger, M., Oberguggenberger, M., Steinbauer, R.: Geometric Theory of Generalized Functions with Applications to General Relativity. In particular chapter 3.1

Talk 1.3: Distributional Energy Conditions & Focusing

Abstract: This talk will introduce the distributional strong energy condition and the distributional null energy condition. We will see that these conditions are equivalent to the usual definitions if the metric is at least C^2 and show that these imply "surrogate" energy conditions for suitable nearby smooth metrics which are strong enough to allow one to prove focusing statements for these nearby smooth metrics.

Literature: Section 4.4 of [S], respectively Section 4.1 plus Lem. 4.10 and Section 5 until (incl.) Lem. 5.6 of [G];

Talk 1.4: Geodesics and the proof of the first C^1-theorems

Abstract: The goal of this fourth talk will be to finally prove one (or both) of the C^1 singularity theorems of [G]. However before proceeding with the proof there is a final missing piece of background material to be covered, namely properties (and definitions) of geodesics for C^1-metrics.

Literature: Section 4.6 of [S], respectively Sections 2.1, 2.2, and Thm. 4.11, 4.13 and Thm. 5.7 of [G]

Talk 1.5: The Hawking-Penrose Theorem

Abstract: Sometimes referred to as the "singularity theorem par excellence" the Hawking-Penrose theorem is significantly more involved, essentially combining both previous results into a single theorem and replacing the strong assumption of global hyperbolicity with a genericity condition on the Riemann tensor. This talk would involve discussing the distributional version of the genericity condition and proving a no lines statement for C^1 metrics from there.

Literature: A good starting point is [S], from p. 89 to the end of Section 3 and from the end of Section 4.6 to Section 4.10, mainly focusing on 4.7 and 4.8. Other sources include

- [KOSS] and perhaps also [GGKS] Graf, Grant, Kunzinger, steinbauer: The Hawking-Penrose singularity theorem for $C^{1,1}$-Lorentzian metrics `https://arxiv.org/abs/1706.08426`

- [BEE], Chapter 12.4 and/or [HE] Hawking, Ellis: The Large Scale Structure of Space-Time, esp. Section 8, Thm. 2; for background on the classical Hawking-Penrose theorem

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Talk 1.6 (by the mentor): Outlook and related topics

Abstract: The exact content depends on the previous talks and the interests of the participants. I will talk about other results in the literature (connected to the singularity theorems or low regularity more generally) and different approaches to low regularity Lorentzian geometry (such as Lorentzian length spaces), open questions and current research.

Stephen McCormick: Notions of mass in General Relativity

Overall plan: From the perspective of general relativity, one cannot speak directly about the gravitational field or the mass/energy that it carries. However, there are well-defined notions of the total mass of an isolated gravitating system, and some attempts to (quasi-)localise it. In the case of a vanishing cosmological constant, initial data for an isolated gravitating system is given by an asymptotically flat/Euclidean Riemannian manifold (sometimes equipped with additional tensor fields). The total mass of such a system can be defined via a Hamiltonian (the ADM mass) and this turns out to be intimately connected to scalar curvature. For this reason, the study of mass in general relativity is intimately connected with the study of manifolds with positive scalar curvature. These lectures will focus on mass from this geometric analytic perspective, and aims to be at an introductory level to some of the main ideas surrounding the topic.

We will focus predominantly on what is called the *time-symmetric* situation, where more is understood and the connection to scalar curvature is more direct. The aim is to cover the major results for the mass of an asymptotically flat manifold, and the later lectures are somewhat steered by my own interest in the problem of quasi-local mass. The view of mass from the perspective of a geometric analyst has been covered quite well by Dan Lee's relatively recent book "*Geometric Relativity*", which will be a useful reference for all of the lectures. The proposed lectures have some flexibility/freedom for the speakers to choose which direction to take and what aspects to emphasise over others. So it is possible make changes based on the interests of the speakers, and I'm more than happy to provide guidance on different directions each talk could take.

Talk 2.1: Hamiltonian formulation of general relativity and the ADM mass

Abstract: This lecture will introduce the the initial data formulation of general relativity and derive the ADM Hamiltonian formulation. This leads to the introduction of the so-called Regge-Teitelboim Hamiltonian, from which the ADM energy and linear momentum can be read off. We see that Hamilton's equations coincide with the Einstein evolution equations, and in particular that elements of the kernel of the adjoint of the linearised constraint map correspond to Killing vectors in the spacetime.

Literature:

(1) Gourgoulhon, E. "3+ 1 formalism and bases of numerical relativity." arXiv preprint gr-qc/0703035 (2007).
(2) Regge, T. and Teitelboim, C. "Role of surface integrals in the Hamiltonian formulation of general relativity." Annals of physics 88.1 (1974): 286-318.
(3) Bartnik, R. "Phase space for the Einstein equations." Communications in Analysis and Geometry 13.5 (2005): 845-885. [Section 5]
(4) Moncrief, V. "Space–time symmetries and linearization stability of the Einstein equations. II." Journal of Mathematical Physics 17.10 (1976): 1893-1902.
(5) The original work of Arnowitt, Deser, and Misner (ADM) may also be a useful source, however it is developed over the course of several years and I believe more modern sets of lecture notes will be easier to follow.

The lecture notes by Gourgoulhon are the standard reference for an introduction to the topic, however there are now many other sets of lecture notes by different authors that could be used.

Talk 2.2: The positive mass theorem

Abstract: This lecture introduces the positive mass theorem with some background and special cases before providing an outline Schoen and Yau's classical proof of the Riemannian positive mass theorem from a modern geometric perspective. If time permits, the general spacetime (non-Riemannian) case will be discussed.

Literature:

(1) Schoen, Richard, and Shing-Tung Yau. "On the proof of the positive mass conjecture in general relativity." Communications in Mathematical Physics 65 (1979): 45-76.
(2) Schoen, Richard, and Shing-Tung Yau. "Proof of the positive mass theorem. II." Communications in Mathematical Physics 79 (1981): 231-260.

The book *Geometric Relativity* by Dan Lee, mentioned above, has a very nice exposition and following this and references therein would be a good starting point.

Talk 2.3: The Riemannian Penrose Inequality

Abstract: The Penrose inequality is briefly introduced and motivated by physical heuristics before a proof of the Riemmanian case is outlined. The speaker will have discretion to follow the proof of either Huisken - Ilmanen or that of Bray, depending on their own background and interests. Both proofs rely on geometric flows in different ways. If time permits, some discussion of strengthened versions of the Riemannian Penrose inequality will be given.

Literature:

(1) Mars, M. "An overview on the Penrose inequality." Journal of Physics: Conference Series. Vol. 66. No. 1. IOP Publishing, 2007.
(2) Mars, M. "Present status of the Penrose inequality." Classical and Quantum Gravity 26.19 (2009): 193001.
(3) Huisken, G. and Ilmanen, T. "The inverse mean curvature flow and the Riemannian Penrose inequality." Journal of Differential Geometry 59.3 (2001): 353-437.
(4) Bray, H. "Proof of the Riemannian Penrose inequality using the positive mass theorem." Journal of Differential Geometry 59.2 (2001): 177-267.

As with the positive mass theorem, Dan Lee's book has an exposition on the Riemannian Penrose inequality (Chapter 4), including an overview of both proofs.

Talk 2.4: Quasi-local mass I: The Bartnik mass

Abstract: We begin by discussing the problem of quasi-local mass and the properties one should expect a good definition to satisfy, including some mention of the various attempts throughout the literature. Then we turn focus to the definition due to Bartnik to examine it in the context of the properties a good definition should satisfy. The related static metric extension conjecture is also discussed, and makes connection with Lecture 1.

Literature:

(1) Bartnik, R. "New definition of quasilocal mass." Physical review letters 62.20 (1989): 2346.
(2) McCormick, S. "An Overview of Bartnik's Quasi-Local Mass." arXiv preprint arXiv:2401.05128 (2024) textit(and references therein)
(3) Szabados, L. "Quasi-local energy-momentum and angular momentum in GR: a review article." Living reviews in relativity 7 (2004): 1-140.

This is a little vague and the speaker is encouraged to choose different properties to emphasise or proofs to sketch. Depending on this, different references will be more valuable so the speaker is encouraged to look at the references cited in my overview article for motivation. The review article by Szabados covers most quasi-local mass definitions that have been defined, and should only be used to give the speaker some general background.

Talk 2.5: Quasi-local mass II: The Brown-York and Wang-Yau masses

Abstract: This lecture will focus on the Brown-York mass both from the perspective of an interesting geometric quantity, and as an entry point to the Wang-Yau mass. The mass is introduced from a quasi-local Hamiltonian perspective, naturally leading to the large sphere limit recovering the ADM mass. The proof of the positivity of the Brown-York mass due to Shi and Tam is then sketched, before concluding with some comments regarding the more recently defined Wang-Yau mass.

Literature:

(1) Brown, D. and York, J. "Quasilocal energy and conserved charges derived from the gravitational action." Physical Review D 47.4 (1993): 1407.
(2) Gourgoulhon, E. "3+ 1 formalism and bases of numerical relativity." arXiv preprint gr-qc/0703035 (2007). - *These can be used as a reference for the Hamiltonian formulation, to connect to Lecture 1*
(3) Shi, Y., and Tam, L.-F. "Positive mass theorem and the boundary behaviors of compact manifolds with nonnegative scalar curvature." Journal of Differential Geometry 62.1 (2002): 79-125.
(4) Wang, M.-T., and Yau, S.-T. "Quasilocal mass in general relativity." Physical review letters 102.2 (2009): 021101.

The Shi-Tam paper is where the technical aspects of the talk lie.

Talk 2.6 (by the mentor): Survey and Outlook

Abstract: In this lecture, I will outline a selection of recent results and active work in the general field of mass in general relativity.

Sam C. Collingbourne: Stability and Instability of Black Holes in General Relativity

Overall plan: Famously, the vacuum Einstein equation admits black hole solutions, the simplest of which was written down by Schwarzschild mere months after Einstein proposed the theory in 1915. A black hole is characterised by two regions, an exterior and an interior, which are separated by an event horizon. The interior is 'causally disconnected' from the exterior, meaning not even light can escape, with the event horizon being the surface of no return (to the exterior).

A important strand of research in general relativity is to investigate the stability of black hole exteriors since this predicates the existence of such objects in astrophysics. In view of the well-posedness of the initial value problem, one can ask what happens when we perturb initial data for a known solution. Stability is expected to be true of the exterior of all 4-dimensional, stationary, vacuum black holes that form the Kerr family of rotating black holes. This series of talks will investigate some key ideas/methods in the story of proving that the Kerr family of black holes are stable. There will also be some talks looking at other settings where instabilities arise and what is expected in higher dimensions.

Talk 3.1: Introduction to 4D Vacuum Black Holes and the Equations Governing Linear Perturbations

Abstract: This talk will give the basic setup for the stability problem, review the Kerr black hole, and discuss the equations governing linear perturbations. In particular, the geometric structure of Kerr and Schwarzschild that plays a important role in studying stability questions will be reviewed, i.e. the redshift effect, geodesic stucture (trapping in Schwarzschild & Kerr), and the ergoregion (in Kerr). Additionally, this talk will discuss the linearised Einstein equation, the Teukolsky wave equation and the Chandraskhar transformation to the Regge-Wheeler wave equation.

Literature:

- H. Reall, "Part III General Relativity" section 12 (section 10 is instructive for deriving the linearised Einstein equation). `https://www.damtp.cam.ac.uk/user/hsr1000/lecturenotes_2012.pdf`
- H. Reall, "Part III Black Holes" section 7.`https://www.damtp.cam.ac.uk/user/hsr1000/black_holes_lectures_2016.pdf`
- S. Aretakis "Dynamics of Extremal Black Holes" sections 1.2-1.7`https://link.springer.com/book/10.1007/978-3-319-95183-6`.
- M. Dafermos and I. Rodnianski, "Lectures on black holes and linear waves" sections 1 and 2.`https://www.claymath.org/library/proceedings/cmip017c.pdf`
- M. Dafermos, I. Rodnianski and G. Holzegel "The linear stability of the Schwarzschild solution to gravitational perturbations" sections 2.1.1-2.1.5.`https://arxiv.org/pdf/1601.06467`

Comment: The Teukolsky equation on Kerr should be mentioned, but no detailed discussion of it's derivation need be provided: see Teukolsky's 1973 or Ryan's 1974 papers for interest.

Talk 3.2: The Vector Field Method in General Relativity

Abstract: This talk will focus on the "vector field method" for the wave equation on the Minkowski and Schwarszchild, where one uses the energy momentum tensor, suitably chosen vector fields and the divergence theorem to obtain energy estimates. In particular, this talk will discuss the key ingredients for obtaining energy boundedness and decay for the wave equation: the setup for the vector field method, conservation laws and spacetime estimates such as the Morawetz or r^p estimate. Time permitting this talk can discuss the extra ingredients/obstacles for the Schwarzschild spacetime (the redshift effect and trapping at the photon sphere) or how these ideas can be applied for linearised gravity (see the Teukolsky equation and Chandrasekhar transformation above).

Literature:

- S. Aretakis "Lecture Notes on General Relativity" sections 9.1, 9.2 and 9.4 and section 10.`https://web.math.princeton.edu/~aretakis/columbiaGR.pdf`.
- G. Holzegel "General Relativity and the Analysis of Black Hole Spacetimes" section 4.1-4.4 and 5.1-5.4.`https://www.uni-muenster.de/IVV5WS/WebHop/user/gholzege/Skript%20WS%202021%202022.pdf`
- M. Dafermos and I. Rodnianski, "Lectures on black holes and linear waves". `(https://www.claymath.org/library/proceedings/cmip017c.pdf`

Talk 3.3: Mode Stability for the Wave/Teukolsky Equation on Kerr

Abstract: This talk will discuss the first check in understanding stability for Kerr (or, indeed, any spacetime), whether there exist any exponentially growing solutions to the wave equaton or the equations of linearised gravity. This talk will discuss the simple proof of mode stability for the Schwarzschild spacetime and review Whiting's or Shlapentokh-Rothman's proof for Kerr using an integral transformation for the radial Teukolsky ODE (one could alternatively discuss the method Casals-Teixeira da Costa via the continued fraction method).

Literature:

- Y. Shlapentokh-Rothman, "Quantitative Mode Stability for the Wave Equation on the Kerr Spacetime" `https://arxiv.org/pdf/1302.6902)`
- R. Teixeira da Costa, "Mode stability for the Teukolsky equation on extremal and subextremal Kerr spacetimes" `https://arxiv.org/abs/1910.02854`.
- B. Whiting, "Mode stability of the Kerr black hole", J. Math. Phys. 30 (1989), no. 6, 1301-1305.
- M. Casals and R.Teixeira da Costa, "Hidden spectral symmetries and mode stability of Kerr(-dS) black holes" section 2 `https://arxiv.org/abs/2105.13329`.

Talk 3.4: The Superradient Instability of the Klein-Gordon Equation on Kerr

Abstract: Whilst Kerr is expected to be stable as a solution to the *vacuum* Einstein equation, as a solution to the Einstein-Klein-Gordon system the Kerr black hole is expected to be unstable. This talk will review the paper of Shlapentokh-Rothman on the instability that arises for the Klein-Gordon equation.

Literature:

- Y. Shlapentokh-Rothman, "Exponentially growing finite energy solutions for the Klein-Gordon equation on sub-extremal Kerr spacetimes", `https://arxiv.org/abs/1302.6902`.
- O. Chodosh and Y. Shlapentokh-Rothman, "Time-Periodic Einstein–Klein–Gordon Bifurcations of Kerr", section 1 `https://arxiv.org/pdf/1510.08025`.

Talk 3.5: The Aretakis Instability of the Wave Equation on Extremal Reissner-Nordström

Abstract: Reissner-Nordström is a spherically symmetric massive, charged black hole solution to the Einstein-Maxwell system. It is often viewed as a toy model for Kerr; "charge, Q, is a poor man's angular momentum'. To exhibit a black hole spacetime one has the restriction $Q \leq M$, where M is the mass of the black hole, with equality denoted as *extremal* Reissner-Nordström. This talk will focus on the horizon conservation laws for the wave equation established by Aretakis in his semial work. The talk will discuss how this gives rise to blow up of transversal derivatives along the horizon.

Literature:

- S. Aretakis "Dynamics of Extremal Black Holes" sections 1.5-1.7 and 2.1-2.2 (one could read further in section 2) `https://link.springer.com/book/10.1007/978-3-319-95183-6`
- S. Aretakis "Stability and Instability of Extreme Reissner–Nordström Black Hole Spacetimes for Linear Scalar Perturbations II" sections 1-3 (this is more technical) `https://arxiv.org/abs/1110.2009`.

Talk 3.6 (by mentor): Stability and Instability for Higher-Dimensional Black Holes

Abstract: Stability problems in higher dimensions are largely open. In this talk, I will discuss expectations of stability in higher dimensions. I will discuss the wave equation on higher dimensional Schwarzschild, Gregory-Laflamme instability, the failure of the Teukolsky/Regge-Wheeler approach and a possible resolution to obtain stability statements for linearised gravity on higher dimensional Schwarzschild.
